

PRESERVING AND APPROXIMATING EFFICIENT SOLUTIONS IN MULTIOBJECTIVE OPTIMIZATION UNDER FUNCTION UNCERTAINTY

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ABSTRACT. Many engineering systems are designed to satisfy multiple and conflicting objectives while their performance with respect to these objectives is uncertain. Motivated by such design problems, we develop a model of uncertain multiobjective optimization problem (UMOP) under function uncertainty and present two robust counterparts whose efficient solutions are robust efficient for the UMOP. While the counterparts account for all, possibly infinitely many scenarios, their efficient sets are preserved in or approximated by the efficient sets of associated finite MOPs. The properties of preservation and approximation require that the uncertain functions be constructed by order-preserving operators applied to basis functions making up the objective functions of the finite MOPs. The proposed model of UMOPs under function uncertainty and its resolution are applicable to monotone and linear systems in science and engineering because their performance is governed by order-preserving operators. Examples of linear operators making up a big class of order-preserving operators are included. An engineering design problem illustrates the UMOP model and solution approaches.

Keywords. Robust efficient solutions, Pareto-optimal set, nondominated set, order-preserving operators.

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1. INTRODUCTION

Computing efficient sets for multiobjective optimization problems (MOPs) has been well studied and is known to be difficult for some classes of problems (see [17] among others). The computational difficulty grows with an increase in the number of objective functions. The effects of adding new objective functions are discussed in [30] and [55].

The presence of many criteria in the MOP naturally results from the real-life decision-making situation being modeled. Such models include 4-objective problems in engineering [31] and radiotherapy [79], 9-objective control problem [9], 7-objective classification problem [53], and 10-objective calibration problem in ecology [62]. A wide range of applications from aerodynamic design to medical decision making and land use planning are presented in [70] with the number of objectives up to hundreds.

Another reason for MOPs with many criteria comes from modeling uncertain single objective optimization problems (USOPs) as MOPs because every objective function may reflect a different uncertainty realization in the USOP [35, 50, 49, 41, 59]. In particular, infinite scenario sets leading to infinite MOPs, that is, MOPs with infinitely many objective functions, are studied in [49]. Theoretical studies on infinite MOPs are limited. [77] and later [25] extend the concept of Geoffrion properly efficient solutions [32] for such problems. However, infinite MOPs are applied to solve set optimization problems

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where the original problem is replaced with an infinite MOP that is later made finite and therefore numerically tractable [23, 22].

Uncertainty may naturally be associated with MOPs in many ways depending on its source and the part of the MOP that carries it, and may lead to different models and approaches within a robust multiobjective optimization paradigm. Staying within deterministic modeling of uncertainty, two types of uncertainties are common: *parameter uncertainty* assuming uncertain coefficients of the objective or constraint functions (see the bibliographical sources referred to throughout the paper whenever suitable), and *decision uncertainty* meaning the decision being made may be uncertain [20]. Refer to [44] for a survey of robustness concepts and to [75] for a tutorial on different types of uncertain MOPs (UMOPs). In the latter, still a different model is included, an UMOP with *uncertain objective functions*. This type of UMOPs has not been studied elsewhere and is of interest in this paper.

Theoretical studies on UMOPs have been accompanied by computational methods developed within the areas of parametric or robust multiobjective optimization. Within parametric optimization, path following methods [13] and variational methods [78] are available for unconstrained nonlinear single-parametric problems, while exact and approximate algorithms [15, 47] are available for linear and convex quadratic multiparametric problems with polyhedral uncertainty sets. Only the former have been applied to UMOPs with more than two objectives. In robust settings, solution methods have been developed for biobjective linear problems with almost 5000 variables under discrete uncertainty modeled by 4000 parameters [14], biobjective nonconvex problems with two variables and convex decision uncertainty sets [21], and multiobjective convex quadratic problems with up to five objectives, 1000 variables, and five parameters drawn from ellipsoidal uncertainty sets [11]. While the scope of all these studies is broad, they do not address a variety of UMOPs with many convex or nonconvex criteria.

The objective of this work is to address UMOPs leading to computationally challenging optimization problems when solved in parametric or robust settings. The approach is similar to that used in set optimization, that is, the UMOP is associated with a finite MOP whose efficient solutions are related to the robust efficient solutions to the original UMOP. The study is motivated by engineering or industrial problems that are exemplified by a design problem described below.

1.1. Motivation from engineering design. We present a decision-making situation from the area of engineering design that commonly uses multiobjective models. In order to realistically model the design process, the designer has to account for infinitely many realizations in which the object being designed will perform. These realizations imply the existence of infinitely many functions modeling the object performance. Truss design problems are structural optimization problems that are commonly used as examples in the engineering optimization literature to verify the effectiveness and usability of new approaches and methods [61]. In this study, the four-bar cantilever truss under load uncertainty is selected as a representative problem. Simplified for computational efficiency, this topology acts as a fundamental engineering substructure, analogous to critical components in large-scale systems, such as a crane jib or an aircraft wing rib. Using this truss problem allows for a demonstration of the UMOP that is focused on robustness under the given uncertain loading without the computational expense of full-scale models.

Figure 1 depicts a four-bar truss, a structure that consists of four bars made of a certain material and is subjected to various loads applied at different points. The lengths of the bars are known but their cross-sectional areas have to be determined in order to guarantee that the truss will perform well in different loading scenarios. When a variety of forces with different directions and magnitudes are applied to the truss, the structure carries stress and may deflect at its joints. The quality of the truss is evaluated by measuring its overall weight, and the vertical deflection and stress at the joints. It is desirable that the truss weight be low, the deflections be small, and the stress be below a permissible value.

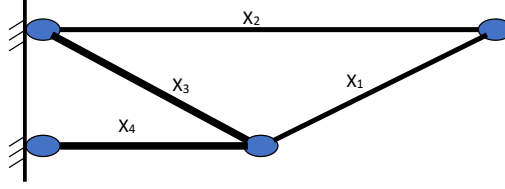


FIGURE 1. 4-bar truss

The four-bar truss design problem first presented in [51, 52] has been used in the engineering design and operations research communities to demonstrate research findings [68, 28, 26] and has also served as an inspiration for this study. To design a n -bar truss, the designer makes a decision about the cross sectional area of each bar $x_j \in \mathbb{R}$, $j \in \{1, \dots, n\}$, which determines the truss geometry $\mathbf{x} \in \mathbb{R}^n$, where $\mathbb{R}^n$ is a set of n -dimensional real vectors. All cross sectional areas that lie within some upper and lower bounds and satisfy the stress constraint for each bar determine the feasible set of designs, $X \subset \mathbb{R}^n$. The truss weight is represented by its volume that is a function of the geometry. The deflection is not only a function of the geometry, $\mathbf{x}$, but also of the forces, $\mathbf{F}$, generated by the applied loads. Clearly, the truss may be subjected to infinitely many different loads that generate many forces acting on the truss. Hence, this design problem can be formulated as an uncertain biobjective optimization problem (UBOP)

$$\left\{ \min_{\mathbf{x} \in X} [\text{volume}(\mathbf{x}) \quad \text{deflection}(\mathbf{x}, \mathbf{F})]^T \right\}_{\mathbf{F} \in \mathcal{F}} \quad (1.1)$$

in which the weight and deflection are minimized with respect to the truss geometry subject to the feasible set X and the set $\mathcal{F}$ of forces $\mathbf{F}$ that can act on the truss due to a variety of loading configurations. In practice, $\mathcal{F}$ is typically a discrete set but it is of interest to make it continuous. Designing the truss corresponds to solving this UBOP, that is, finding efficient cross sectional areas for infinitely many realizations of the forces in $\mathcal{F}$, where $\mathcal{F}$ plays the role of an uncertainty set associated with this design problem.

UBOP (1.1) carries parameter uncertainty and one would expect that this problem could be solved for the case of four bars with a parametric or robust multiobjective optimization algorithm. However, the current state-of-the-art algorithms do not offer a suitable tool for this rather simple case not to mention trusses with 10 or more bars, many loads, and boundary conditions. In fact, larger truss design problems even with no uncertainty are already computationally challenging and require special numerical methods (e.g., the finite element method [48]). Adding uncertainty makes such design problems challenging and makes heuristic algorithms or simulation the only tools for optimization [58].

Trusses are examples of many real-life systems encountered in electrical engineering, physics, geology, process control, computing, and other disciplines, which are known as linear systems. They act in agreement with the superposition principle according to which the system output resulting from two or more inputs is the sum of the responses that would be caused by each input individually. In mathematical terms, this principle is defined by linear functions having the properties of additivity and homogeneity.

1.2. Research objective. In this paper, we address decision-making problems that are characterized by multiple criteria and perform under uncertainty. We offer an interdisciplinary perspective making use of concepts from multiobjective optimization, algebra, and engineering design. While we build on some well-established results, we introduce the concept of *function uncertainty*. We motivate the concept of uncertainty sets constructed by functions and develop a modeling methodology to relate the UMOP with function uncertainty to deterministic and finite MOPs whose efficient solutions, under certain conditions, are robust efficient to the UMOP. In some cases the relationship between the efficient

set and the robust efficient set is very strong allowing for the preservation of the latter by the former. The conditions do not require convexity of the objective functions and no restriction is placed on the feasible region. However, the uncertain functions have to be generated by order-preserving operators. Indeed, such operators govern the performance of monotone or linear systems in many areas of engineering and science. Examples of monotone systems can be found in biochemical networks, power flow networks, models of fluid dynamics [67], while linear systems are commonly encountered in structural analysis, circuit design, control engineering, and signal processing. A large class of order-preserving operators is made up by linear operators, which are mathematical models of performance of linear systems governed by the principle of superposition.

The paper is organized into eight sections. Starting with the well-known parameter uncertainty in Section 2, we develop a model of the UMOP with function uncertainty. In the subsequent two sections, we propose two multiobjective counterparts to the UMOP whose efficient sets, being robust efficient to the UMOP, are preserved or approximated by the efficient sets of associated deterministic and finite MOPs. These models and approaches are compared with each other, and their applicability is contrasted with that of the UMOP with parameter uncertainty in Section 5. In Section 6, examples of functions satisfying the conditions for the preservation or approximation are given. The robust efficient solutions for the engineering design problem presented above are computed in Section 7 and the paper is concluded in Section 8.

2. TOWARD FUNCTION UNCERTAINTY

Let $b, n, p, q, \theta \in \mathbb{N}$. We denote by $[b]$ the set of the first b positive integers, that is, $[b] := \{1, \dots, b\}$ for every positive integer b . Let $X \subseteq \mathbb{R}^n$ be a feasible set, and $\psi_i : \mathbb{R}^n \times \mathbb{R}^{\theta_i} \rightarrow \mathbb{R}$ be scalar-valued objective functions defined on the decision space $\mathbb{R}^n$ and uncertainty space $\mathbb{R}^{\theta_i}$ for every $i \in [q]$. Consider a UMOP with parameter uncertainty as a collection of MOPs

$$\left\{ \min_{\mathbf{x} \in X} [\psi_1(\mathbf{x}, \boldsymbol{\xi}_1) \quad \dots \quad \psi_q(\mathbf{x}, \boldsymbol{\xi}_q)] \right\}_{(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_q) \in \Xi_1 \times \dots \times \Xi_q} \quad (2.1)$$

where $\Xi_i \subset \mathbb{R}^{\theta_i}$, for every $i \in [q]$, are nonempty, not necessarily finite, sets of uncertain parameters affecting the objective values. The model of parameter uncertainty, described in (2.1), is referred to as *objective-wise uncertainty* and was first considered in [54] and [18]. In the latter, this model is used to define so-called robust efficient solutions for MOPs under parameter uncertainty and robust versions of the weighted-sum and ϵ -constraint scalarizations as well as the objective-wise worst case MOP are presented. Efficient solution concepts and their properties for MOPs with uncertain parameters in the objective functions and the partial order governing the MOP are studied in [36]. The uncertainty set consists of countably many, but possibly infinitely many scenarios.

For every $i \in [q]$, we assume that the objective function ψ_i is known for the values of the uncertain parameter, $\boldsymbol{\xi}_i$, that are assumed in b_i primary scenarios or realizations, $\boldsymbol{\xi}_i^1, \dots, \boldsymbol{\xi}_i^{b_i}$, that is, $v_{i1}(\mathbf{x}) := \psi_i(\mathbf{x}, \boldsymbol{\xi}_i^1), \dots, v_{ib_i}(\mathbf{x}) := \psi_i(\mathbf{x}, \boldsymbol{\xi}_i^{b_i})$ are known. Overall, there are $b = \sum_{i=1}^q b_i$ primary scenarios. Note that for $q = 1$ this modeling approach agrees with the earlier discussed research direction of converting a USOP into a finite MOP. The finite MOP obtained for (2.1) is

$$\min_{\mathbf{x} \in X} \mathbf{v}(\mathbf{x}), \quad (2.2)$$

where $\mathbf{v} : \mathbb{R}^n \rightarrow \mathbb{R}^b$ is a vector-valued objective function. MOP (2.2) is finite (i.e., it has finitely many objective functions) and does not carry the same information as UMOP (2.1). Before we address this issue, we recall the known solution concepts that can be applied to MOP (2.2).

Given vectors $\mathbf{z}, \mathbf{z}' \in \mathbb{R}^b$, we have $\mathbf{z} < \mathbf{z}' \Leftrightarrow z_j < z'_j$ for every $j \in [b]$, $\mathbf{z} \leq \mathbf{z}' \Leftrightarrow z_j \leq z'_j$ for every $j \in [b]$ and $\mathbf{z} \neq \mathbf{z}'$, and $\mathbf{z} \leq \mathbf{z}' \Leftrightarrow z_j \leq z'_j$ for every $j \in [b]$. It is to be noted that if $b = 1$, $\mathbf{z}(\leq) \leq \mathbf{z}'$ is equivalent to the scalar comparison $z(\leq) < z'$.

With these relations in mind, a solution $\mathbf{x} \in X$ is called *(weakly) efficient* to MOP (2.2) if there does not exist $\bar{\mathbf{x}} \in X$ such that $\mathbf{v}(\bar{\mathbf{x}})(<) \leq \mathbf{v}(\mathbf{x})$. The set of (weakly) efficient solutions to MOP (2.2) is denoted $(E_w(X, \mathbf{v})) E(X, \mathbf{v})$. A solution $\mathbf{x} \in X$ is called *strictly efficient* to MOP (2.2) if there does not exist $\bar{\mathbf{x}} \in X, \bar{\mathbf{x}} \neq \mathbf{x}$, such that $\mathbf{v}(\bar{\mathbf{x}}) \leq \mathbf{v}(\mathbf{x})$. The set of strictly efficient solutions to MOP (2.2) is denoted $E_s(X, \mathbf{v})$ [60, 17]. We have $E_s(X, \mathbf{v}) \subseteq E(X, \mathbf{v}) \subseteq E_w(X, \mathbf{v})$. The set of efficient, strictly efficient, or weakly efficient solutions to MOP (2.2) is denoted $E_{(\cdot/s/w)}(X, \mathbf{v})$. Similarly, the set of efficient or strictly efficient solutions to MOP (2.2) is denoted $E_{(\cdot/s)}(X, \mathbf{v})$. We assume the existence of the efficient set of problem (2.2) and the availability of methods to compute this set [73].

To assure that MOP (2.2) represents all functional values in (2.1) resulting from uncertainties $\xi_i \in \Xi_i$, $i \in [q]$, we augment it with an appropriate uncertainty set. Assume that these uncertain values are collected in a nonempty set $\mathcal{U}$ constructed as

$$\mathcal{U} := \{\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^p, \mathbf{f} = \mathbf{h} \circ \mathbf{v}, \mathbf{h} \in \mathcal{H}\} = \{\mathbf{f}^k : \mathbb{R}^n \rightarrow \mathbb{R}^p, \mathbf{f}^k = \mathbf{h}^k \circ \mathbf{v}, k \in \mathcal{K}\}, \quad (2.3)$$

where $\mathcal{H} := \{\mathbf{h}^k, k \in \mathcal{K}\}$ is a set of operators $\mathbf{h}^k : \mathbb{R}^b \rightarrow \mathbb{R}^p$, $k \in \mathcal{K}$, $\mathcal{K}$ is an index set that may be uncountable, $p \geq 1$, and $\mathbf{v} := [v_1 \ \dots \ v_b]^T$ is the vector-valued function modeling the b scenarios.

We consider two types of uncertainty sets $\mathcal{U}$ depending on how the operators of $\mathcal{H}$ map vector $\mathbf{v}$ whose components are referred to as basis functions.

In the first type, called *vector-valued uncertainty*, the uncertain objective functions interact with each other through the entire basis vector-valued function $\mathbf{v}$, that is, the component functions of the vector-valued operators of $\mathcal{H}$ may involve basis functions associated with different objective functions of (2.1). Such uncertainty set therefore is not objective-wise as the basis functions are mixed together independently of the objective functions they are associated with. To maintain the association between the objective functions of (2.1) and their basis functions, we introduce another type of uncertainty.

The second type, called *scalar-valued uncertainty*, utilizes the same vector-valued functions of $\mathcal{H}$ but in a different way based on a partition of basis vector-valued function $\mathbf{v} : \mathbb{R}^n \rightarrow \mathbb{R}^b$ into q vector-valued functions $\mathbf{v}_i : \mathbb{R}^n \rightarrow \mathbb{R}^{b_i}$. Each basis vector-valued function $\mathbf{v}_i$ is made up of the b_i basis scalar-valued functions $v_{i1}, \dots, v_{ib_i}$ associated with i th objective function of (2.1), $i \in [q]$. For every operator $\mathbf{h} \in \mathcal{H}$, each component function corresponds to a scalar-valued operator $h_i : \mathbb{R}^{b_i} \rightarrow \mathbb{R}$, $i \in [q]$, that maps basis vector-valued function $\mathbf{v}_i$. For every $i \in [q]$, let $\mathcal{H}_i := \{h_i^{k_i}, k_i \in \mathcal{K}_i\}$ be composed of all the i th component functions of operators in $\mathcal{H}$, where $k_i \in \mathcal{K}_i$ and $\mathcal{K}_i$ is an index set that may be uncountable. Note that with scalar-valued uncertainty, we must have $p = q$. For every $i \in [p]$, the scalar-valued functions of $\mathcal{H}_i$ generate the uncertainty set

$$\mathcal{U}_i := \{f_i : \mathbb{R}^n \rightarrow \mathbb{R}, f_i = h_i \circ \mathbf{v}_i, h_i \in \mathcal{H}_i\} = \{f_i^{k_i} : \mathbb{R}^n \rightarrow \mathbb{R}, f_i^{k_i} = h_i^{k_i} \circ \mathbf{v}_i, k_i \in \mathcal{K}_i\}, \quad (2.4)$$

that is, set $\mathcal{U}_i$ contains a possibly infinite number of realization of the i th objective function of (2.1), but may not contain the basis functions $v_{i1}, \dots, v_{ib_i}$. Uncertainty set (2.3) can now be written as

$$\mathcal{U} = \mathcal{U}_1 \times \dots \times \mathcal{U}_p, \quad (2.5)$$

and vector-valued objective function as $\tilde{\mathbf{f}}^{\tilde{k}} := [f_1^{k_1} \ \dots \ f_p^{k_p}]^T$ for $\tilde{k} := (k_1, \dots, k_p) \in \tilde{\mathcal{K}} := \mathcal{K}_1 \times \dots \times \mathcal{K}_p$. With scalar-valued uncertainty, each uncertain objective function is defined independently of all the other objective functions and the uncertainty is objective-wise.

With uncertainty set (2.3), we obtain a surrogate UMOP for (2.1) as a collection of MOPs

$$\left\{ \min_{\mathbf{x} \in X} \mathbf{f}(\mathbf{x}) \right\}_{\mathbf{f} \in \mathcal{U}} \quad (2.6)$$

where the uncertain objective functions originate from the set $\mathcal{U}$. For linear objective functions, problem (2.6) necessarily reduces to a UMOP with parameter uncertainty. For $p = 1$, (2.6) is an USOP but having $p > 1$ allows for modeling flexibility reflected in (2.2) meaning that each vector-valued function $\mathbf{f} \in \mathcal{U}$ models the system performance in a collection of scenarios for all objective functions in (2.1).

The set $\mathcal{U}$ is an uncertainty set containing all vector-valued functions that could be generated from the set of primary scenarios, but not necessarily containing them. Considering an applied context and assuming UMOP (2.1) represents the performance of a system, these functions model that system's performance in various scenarios, while the elements of the vector $\mathbf{v}$ represent the system performance that are the most representative for the system or assumed by the system at specific uncertainty scenarios selected by the decision maker. As recognized in [5], uncertainties in engineering systems are due to imprecision in the evaluation of the system output, because models rather than the real physical objects are typically used to perform the evaluation. Hence, the values assumed by the functions in $\mathcal{U}$ can also include measuring and approximation errors resulting from the use of models. Therefore, $\mathcal{U}$ is the system performance space that is spanned by the elements of $\mathbf{v}$ and "filled" with system performance modeled by the operators acting on $\mathbf{v}$. If $\mathcal{U}$ is continuous, the uncertainty set $\mathcal{U}$ shall have infinitely many elements. In an applied setting this may not need to be the case, and $\mathcal{U}$ may have finitely many elements meaning that the system performance space has been discretized.

To illustrate a construction of uncertainty set (2.5) and its relationship with $\Xi_1 \times \dots \times \Xi_q$, we consider problem (2.1) with, for every $i \in [q]$, objective function ψ_i being affine in parameter ξ_i and uncertainty set Ξ_i being polytopic. For every $i \in [q]$, assigning a basis scalar-valued function to each of the b_i extreme points of Ξ_i , we define $\mathcal{H}_i$ as the set composed of the operator $h_i : \mathbb{R}^{b_i} \rightarrow \mathbb{R}$ such that $h_i(\mathbf{v}_i) = \sum_{j=1}^{b_i} \lambda_j v_{ij}$ where λ is any point of the standard $(b_i - 1)$ simplex. It can be seen that for every $i \in [q]$, $\mathcal{U}_i$ is nothing but Ξ_i . This construction, where the extreme points of Ξ_i determine the representative scenarios, is applicable in various areas of human activity.

For example, Internet routing of uncertain traffic [14] may correspond to a possible application of this process. In such optimization problems, one needs to consider traffic uncertainty mainly due to the high difficulty to collect, analyze, and accurately estimate the origin-destination (OD) traffics. Following the polyhedral traffic model proposed in [2], one may specify for every $i \in [q]$, the traffic uncertainty set Ξ as

$$\Xi = \{\xi \in \mathbb{R}^{|\mathcal{D}|} : R\xi \leq \mathbf{d}, \xi \geq \mathbf{0}\},$$

where $\mathcal{D}$ is the set of OD-traffic demands, R is the current (and known) routing matrix of the network, and $\mathbf{d}$ is the vector whose components are the largest observed volume of traffic measured on each arc of the network. Commonly used objective functions are the mean link utilization

$$\psi_1(\mathbf{x}, \xi) = \frac{1}{|A|} \sum_{a \in A} \frac{\xi^T \mathbf{x}_a}{c_a}$$

and the routing cost

$$\psi_2(\mathbf{x}, \xi) = \sum_{a \in A} w_a \xi^T \mathbf{x}_a,$$

where A is the set of network's links and for each link $a \in A$, $\mathbf{x}_a$ is the vector of $|\mathcal{D}|$ flow variables on a , c_a is the capacity installed on a , and w_a is the unitary routing cost on a . Both functions ψ_1 and ψ_2 are linear in ξ and uncertainty set Ξ is a polytope.

As other examples, consider decision situations under uncertainty where some extreme scenarios determine the set of plausible scenarios. In forest management, multiple conflicting objectives such as timber harvest revenue, timber volume, carbon storage, biodiversity indicators are to be maximized under uncertainty resulting from climate changes, natural disasters, and government policies. The uncertainty set may be determined by a combination of extreme scenarios such as the level of greenhouse

gas emissions, the probability of wildfire, or the level of government subsidies. In [64], 12 representative scenarios are developed for forest management. Portfolio optimization is often modeled as a biobjective problem where decisions are made to allocate assets to financial instruments so that two conflicting objective functions, risk and negative return are minimized [57]. Each of these objectives carries uncertainty due to volatility of financial markets caused by various factors such as inflation, interest rate changes, supply chain disturbances, and geopolitical conflicts. In [8], six primary classes of scenarios are identified and further subdivided to provide a universal set from which scenarios of interest can be selected to perform portfolio stress testing. In Section 6 we present linear operators such as convex or conical combinations that can be applied to representative or extreme scenarios to yield an uncertainty set.

We put forward the premise that while problem (2.1) can be challenging to solve or even intractable for some applications, it can be generalized by surrogate problem (2.6). This reformulation converts the original parameter uncertainty set $\Xi_1 \times \dots \times \Xi_q$ into a function uncertainty set $\mathcal{U}$ by means of the discretization of $\Xi_i, i \in [q]$, and the selection of the operators in $\mathcal{H}$ acting on the basis functions in (2.2). We note that the discretization of the parameter space and the creation of the basis functions are always available and can be viewed as a data collection. On the other hand, there exist different ways to define the set $\mathcal{H}$, each one leading to a different uncertainty set $\mathcal{U}$. The relationship between the two uncertainty spaces may remain unknown but the choice of operators seems to affect surrogate problem (2.6). We examine the behavior of (2.6) depending on the properties of the operators in $\mathcal{H}$ and show that some classes of the operators are insignificant and only the data collection phase entirely determines (2.6). Additionally, the process of embedding the original uncertainty set into a function uncertainty space allows to benefit from multiobjective optimization that is the most appropriate tool to handle the multiplicity of functions.

Constructing (2.2) and (2.6) requires care regarding the number of objective functions. MOPs with four or more criteria have been renamed into many-objective optimization problems to recognize the computational and decision-making challenges that are not typical when the number of objectives is lower [29, 24]. Given q objectives in (2.1) and b in (2.2), if every original objective is known in b_i scenarios, it must be that $q < b$. However, to avoid many-objective optimization problems, b shall not be large. In the limiting case of taking one scenario per each original objective, $q = b$. Taking $q > b$ would eliminate some objectives in MOP (2.2) that would not be representative for (2.1). It is also expected that $q = p$, but in any case, the final modeling decision shall depend on the DM.

Before presenting two robust counterparts to (2.6), their efficient sets and their relationships to the efficient set of MOP (2.2), we review the following definitions and properties since they are essential for our work.

Definition 2.1. Let $p \geq 2$. A vector-valued function $\mathbf{h} : \mathbb{R}^b \rightarrow \mathbb{R}^p$ is called *(strictly/strongly) order-preserving* on $\mathcal{Z} \subseteq \mathbb{R}^b$ if for every $\mathbf{z}, \mathbf{z}' \in \mathcal{Z}$, $\mathbf{z}(< / \leq) \leq \mathbf{z}'$, it holds $\mathbf{h}(\mathbf{z})(< / <) \leq \mathbf{h}(\mathbf{z}')$.

Definition 2.2. A scalar-valued function $h_i : \mathbb{R}^{b_i} \rightarrow \mathbb{R}$ is called *(strictly/strongly) increasing* on $\mathcal{Z} \subseteq \mathbb{R}^{b_i}$ if for every $\mathbf{z}, \mathbf{z}' \in \mathcal{Z}$, $\mathbf{z}(< / \leq) \leq \mathbf{z}'$, it holds $h_i(\mathbf{z})(< / <) \leq h_i(\mathbf{z}')$.

For example, the achievement scalarizing functions introduced by [76] are strictly or strongly increasing (see also [17], p. 121). [27] generally define scalarizing functions as (strictly/strongly) increasing functions. In particular, the weighted-sum scalarizing function [32] with nonnegative (and nonzero) weights is strictly increasing and it is strongly increasing with positive weights. The weighted-Chebyshev norm scalarization function [69] is also in this class of functions.

Proposition 2.3. Let $p \geq 2$ and $\mathbf{h} : \mathbb{R}^b \rightarrow \mathbb{R}^p$ be a vector-valued function composed of scalar-valued functions $h_i : \mathbb{R}^{b_i} \rightarrow \mathbb{R}$, $i \in [p]$. Then the function $\mathbf{h}$ is (strictly/strongly) order-preserving if and only if the functions h_i are (strictly/strongly) increasing for every $i \in [p]$.

Proposition 2.4. *Let $p \geq 2$ and $\mathbf{h} : \mathbb{R}^b \rightarrow \mathbb{R}^p$ be a vector-valued function composed of scalar-valued functions $h_i : \mathbb{R}^b \rightarrow \mathbb{R}$, $i \in [p]$. Then the function $\mathbf{h}$ is injective if a function h_i is injective for at least one index $i \in [p]$.*

In the next two sections, we present two approaches, based on different robust counterparts, to handling surrogate UMOP (2.6). Borrowing from the terminology developed within the Multidisciplinary Design Optimization (MDO) community, we refer to the robust counterparts as All-in-One (AiO) and All-at-Once (AaO). MDO has been developed to coordinate results of mathematical modeling and optimization algorithms in various disciplines involved in complex system design (see [12, 71] and the references therein). We find the terms AiO and AaO applicable to our context since they appropriately reflect the type of the reformulations we apply to surrogate UMOP (2.6).

3. ALL-IN-ONE (AiO) APPROACH

The term AiO refers to the original multidisciplinary or distributed design system before it is decomposed into subsystems (see [71] and the references therein). In the AiO approach, we consider a single MOP containing all the realizations of the objective functions

$$\min_{\mathbf{x} \in \mathcal{X}} [\mathbf{f}^k(\mathbf{x})]_{k \in \mathcal{K}}^T, \quad (3.1)$$

to define robust efficient solutions to problem (2.6). The objective function in (3.1) is defined as a column block-vector but the inner transpose operator is omitted for the inner blocks to keep the notation simple. Throughout the paper, this convention is used whenever appropriate.

For the vector-valued uncertainty, let $\mathbf{f}^{\text{v-AiO}}$ denote the objective function of (3.1)

$$\mathbf{f}_{\mathcal{U}}^{\text{v-AiO}} := [\mathbf{f}^k]_{k \in \mathcal{K}}^T.$$

Definition 3.1. Let uncertainty be vector-valued. A solution $\mathbf{x} \in X$ is called *vector-AiO-robust (strictly/weakly) efficient* to UMOP (2.6) if it is (strictly/weakly) efficient to AiO-MOP (3.1). The set of vector-AiO-robust (strictly/weakly) efficient solutions is denoted $E_{(\cdot/s/w)}(X, \mathbf{f}_{\mathcal{U}}^{\text{v-AiO}})$.

As recognized in Section 1, MOPs have been viewed as a result of putting together USOPs under multiple, even infinitely many, scenarios. The AiO model is a multiobjective version of that approach by collecting MOPs under multiple scenarios into one AiO MOP. It is proposed in [1] for uncertain multiobjective linear programs (UMOLPs) with scenario-dependent objective matrices. The AiO model is also developed for a two-stage multiscenario multiobjective decision problem [65] and applied to a supply chain problem in South African sugarcane industry yielding 972 objective functions [66]. The authors claim that this modeling approach is especially suitable to decision problems under deep uncertainty [56].

Finding efficient solutions to AiO-MOP (3.1) might be intractable due to a possibly infinite number of component functions in $\mathbf{f}^{\text{v-AiO}}$. However, we now show that $E_{(\cdot/s/w)}(X, \mathbf{f}_{\mathcal{U}}^{\text{v-AiO}})$ is related to the (strictly/weakly) efficient sets of MOP (2.2). Imposing some condition on the operators in $\mathcal{H}$, we strengthen the general property [27]

$$E(X, \mathbf{v}) \subseteq E_w \left(X, [\mathbf{v} \quad \mathbf{f}_{\mathcal{U}}^{\text{v-AiO}}]^T \right).$$

Proposition 3.2. *Let $\mathbf{u}$ be any subvector of $\mathbf{f}_{\mathcal{U}}^{\text{v-AiO}}$. If the functions in the set $\mathcal{H}$ are*

- (i) *order-preserving, then $E_{(\cdot/s)}(X, \mathbf{v}) = E_{(\cdot/s)} \left(X, [\mathbf{v} \quad \mathbf{u}]^T \right)$,*
- (ii) *strictly order-preserving, then $E_w(X, \mathbf{v}) = E_w \left(X, [\mathbf{v} \quad \mathbf{u}]^T \right)$.*

Proof. (i) We prove the result for efficient solutions. The proof for strictly efficient solutions is analogous. Let $\mathbf{x} \in E(X, \mathbf{v})$ and suppose $\mathbf{x} \notin E(X, [\mathbf{v} \ \mathbf{u}]^T)$. Thus, there exists $\bar{\mathbf{x}} \in X$ such that $[\mathbf{v}(\bar{\mathbf{x}}) \ \mathbf{u}(\bar{\mathbf{x}})]^T \leq [\mathbf{v}(\mathbf{x}) \ \mathbf{u}(\mathbf{x})]^T$. This inequality can hold in two ways: $\mathbf{v}(\bar{\mathbf{x}}) \leq \mathbf{v}(\mathbf{x})$ and $\mathbf{u}(\bar{\mathbf{x}}) \leq \mathbf{u}(\mathbf{x})$, or $\mathbf{v}(\bar{\mathbf{x}}) \leq \mathbf{v}(\mathbf{x})$ and $\mathbf{u}(\bar{\mathbf{x}}) \leq \mathbf{u}(\mathbf{x})$. The former implies that $\mathbf{x} \notin E(X, \mathbf{v})$, a contradiction. So in the latter it must be that $\mathbf{v}(\bar{\mathbf{x}}) = \mathbf{v}(\mathbf{x})$ and therefore $\mathbf{u}(\bar{\mathbf{x}}) = \mathbf{u}(\mathbf{x})$, a contradiction.

Let $\mathbf{x} \in E(X, [\mathbf{v} \ \mathbf{u}]^T)$ and suppose $\mathbf{x} \notin E(X, \mathbf{v})$. Thus, there exists $\bar{\mathbf{x}} \in X$ such that $\mathbf{v}(\bar{\mathbf{x}}) \leq \mathbf{v}(\mathbf{x})$. Since the functions in $\mathcal{H}$ are order-preserving, we have $\mathbf{u}(\bar{\mathbf{x}})(\cdot) \leq \mathbf{u}(\mathbf{x})$, or equivalently $[\mathbf{v}(\bar{\mathbf{x}}) \ \mathbf{u}(\bar{\mathbf{x}})]^T \leq [\mathbf{v}(\mathbf{x}) \ \mathbf{u}(\mathbf{x})]^T$, a contradiction to $\mathbf{x} \in E(X, \mathbf{v})$.

(ii) It is known that the set of weakly efficient solutions to MOP (2.2) is retained when considering additional objective functions. (See Corollary 2.32 in [17].) The second part of the proof follows that of case (i). $\square$

As strongly order-preserving functions also are strictly order-preserving, Proposition 3.2(ii) holds true for the former functions. However, Proposition 3.2(ii) is not valid for order-preserving operators.

It is convenient to retrieve original function $\mathbf{v}$ from the subvector $\mathbf{u}$ of $\mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}}$ while the efficient set does not change. This is guaranteed by the following proposition where an appropriate operator α maps $\mathbf{u}$ to $\mathbf{v}$.

Proposition 3.3. *Let $\mathbf{u}$ be any d -dimensional subvector of $\mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}}$. If there exists an order-preserving operator $\alpha : \mathbb{R}^d \rightarrow \mathbb{R}^b$ such that $\alpha(\mathbf{u}) = \mathbf{v} + \mathbf{c}$ with $\mathbf{c} \in \mathbb{R}^b$, and if the functions in the set $\mathcal{H}$ are*

- (i) *(strongly) order-preserving, then $E_{(\cdot/s)}(X, \mathbf{v}) = E_{(\cdot/s)}(X, \mathbf{u}) = E_{(\cdot/s)}(X, \mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}})$,*
- (ii) *strictly order-preserving, then $E_w(X, \mathbf{v}) = E_w(X, \mathbf{u}) = E_w(X, \mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}})$.*

Proof. We prove the result for efficient solutions. Let $\mathbf{x} \in E(X, \mathbf{u})$ and $\mathbf{x} \notin E(X, \mathbf{v})$. There then exists $\bar{\mathbf{x}} \in X$ such that $\mathbf{v}(\bar{\mathbf{x}}) \leq \mathbf{v}(\mathbf{x})$. Since the functions in $\mathcal{H}$ are order-preserving, we have $\mathbf{u}(\bar{\mathbf{x}}) \leq \mathbf{u}(\mathbf{x})$. If the latter holds as an equation, from the definition of operator α , we obtain $\mathbf{v}(\bar{\mathbf{x}}) = \mathbf{v}(\mathbf{x})$, a contradiction. Therefore, we must have $\mathbf{u}(\bar{\mathbf{x}}) < \mathbf{u}(\mathbf{x})$, a contradiction to $\mathbf{x} \in E(X, \mathbf{u})$. Consequently, $E(X, \mathbf{u}) \subseteq E(X, \mathbf{v})$.

Let next $\mathbf{x} \in E(X, \mathbf{v})$ and $\mathbf{x} \notin E(X, \mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}})$. There then exists $\bar{\mathbf{x}} \in X$ such that $\mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}}(\bar{\mathbf{x}}) \leq \mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}}(\mathbf{x})$. We then have $\mathbf{u}(\bar{\mathbf{x}}) \leq \mathbf{u}(\mathbf{x})$. Since operator α is order-preserving, we obtain $\mathbf{v}(\bar{\mathbf{x}}) + \mathbf{c} \leq \mathbf{v}(\mathbf{x}) + \mathbf{c}$. The efficiency of $\mathbf{x}$ with respect to $\mathbf{v}$ implies $\mathbf{v}(\bar{\mathbf{x}}) = \mathbf{v}(\mathbf{x})$ which yields $\mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}}(\bar{\mathbf{x}}) = \mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}}(\mathbf{x})$, a contradiction. We therefore have $E(X, \mathbf{v}) \subseteq E(X, \mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}})$.

Consider finally $\mathbf{x} \in E(X, \mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}})$ and assume $\mathbf{x} \notin E(X, \mathbf{u})$. There then exists $\bar{\mathbf{x}} \in X$ such that $\mathbf{u}(\bar{\mathbf{x}}) \leq \mathbf{u}(\mathbf{x})$. Since operator α is order-preserving, we obtain $\mathbf{v}(\bar{\mathbf{x}}) + \mathbf{c} \leq \mathbf{v}(\mathbf{x}) + \mathbf{c}$. The order-preserving property of the functions in $\mathcal{H}$ implies $\mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}}(\bar{\mathbf{x}}) \leq \mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}}(\mathbf{x})$. Since $\mathbf{u}$ is a subvector of $\mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}}$ we obtain $\mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}}(\bar{\mathbf{x}}) \leq \mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}}(\mathbf{x})$, a contradiction to $\mathbf{x} \in E(X, \mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}})$. Thus, $E(X, \mathbf{f}_{\mathcal{U}}^{\mathbf{v}-\text{AiO}}) \subseteq E(X, \mathbf{u})$. $\square$

As an example of an operator satisfying the assumption of Proposition 3.3, for $b = p = 3$ define $\mathbf{h}^k \circ \mathbf{v} := e^k \mathbf{v}$, where $\mathbf{v} : \mathbb{R}^n \rightarrow \mathbb{R}^3$ and e^k is the Euler's number raised to the k -th power and $k \in \mathcal{K} = \{0, 1, 2, \dots\}$. Then $\mathcal{U} = \{[v_1 \ v_2 \ v_3]^T, [e v_1 \ e v_2 \ e v_3]^T, [e^2 v_1 \ e^2 v_2 \ e^2 v_3]^T, \dots\}$. We might then consider $\mathbf{u} = [e^{k_1} v_1 \ e^{k_2} v_2 \ e^{k_3} v_3]^T$ with $k_1, k_2, k_3 \in \mathcal{K}$ and operator α be the linear mapping defined by matrix

$$\begin{bmatrix} e^{-k_1} & 0 & 0 \\ 0 & e^{-k_2} & 0 \\ 0 & 0 & e^{-k_3} \end{bmatrix}.$$

Since the diagonal elements of this matrix are positive, operator α is order-preserving. For example, the function $e^{-k} v_i$ may model the exponential decay rate in finance, biology, or physics, where v_i represents an initial quantity.

It is useful to recognize the set of operators satisfying Proposition 3.3.

Definition 3.4. The set $\mathcal{H}$ of functions for which either $E_{(\cdot/s)}(X, \mathbf{v}) = E_{(\cdot/s)}(X, \mathbf{f}_{\mathcal{U}}^{\mathbf{v-AiO}})$ or $E_w(X, \mathbf{v}) = E_w(X, \mathbf{f}_{\mathcal{U}}^{\mathbf{v-AiO}})$ is called *vector-AiO-basis-preserving*.

Proposition 3.3 provides a finite MOP which can be solved instead of UMOP (2.6) to obtain the desired solution set to the latter.

Corollary 3.5. Let $\mathcal{U}$ be vector-valued and set $\mathcal{H}$ be vector-AiO-basis-preserving. A solution $\mathbf{x} \in X$ is vector-AiO-robust (strictly/weakly) efficient to UMOP (2.6) if and only if it is (strictly/weakly) efficient to the finite MOP (2.2).

3.1. Scalar-valued uncertainty. We recognize that the scalar-valued uncertainty can be viewed as a special case of the vector-valued uncertainty as established in (2.5). Therefore, the results presented in this subsection come as corollaries to the propositions previously obtained for the vector approach.

For the scalar-valued uncertainty, let $\mathbf{f}_{\mathcal{U}}^{\mathbf{s-AiO}}$ denote the objective function of (3.1)

$$\mathbf{f}_{\mathcal{U}}^{\mathbf{s-AiO}} := \begin{bmatrix} [f_1^{k_1}]_{k_1 \in \mathcal{K}_1} & \cdots & [f_p^{k_p}]_{k_p \in \mathcal{K}_p} \end{bmatrix}^T.$$

Definition 3.6. Let uncertainty be scalar-valued. A solution $\mathbf{x} \in X$ is called *scalar-AiO-robust* (strictly/weakly) efficient to UMOP (2.6) if it is (strictly/weakly) efficient to AiO-MOP (3.1). The set of scalar-AiO-robust (strictly/weakly) efficient solutions is denoted $E_{(\cdot/s/w)}(X, \mathbf{f}_{\mathcal{U}}^{\mathbf{s-AiO}})$.

Wiecek and Dranichak [75] propose this scalar-AiO robustness concept for MOPs with uncertain objective function coefficients. With set $\mathcal{H}_i := \{h_i^{k_i}, k_i \in \mathcal{K}_i\}$ of operators and basis vector-valued function $\mathbf{v}_i, i \in [p]$, we associate

$$\mathbf{f}_{\mathcal{U}_i}^{\mathbf{s-AiO}} := [f_i^{k_i}]_{k_i \in \mathcal{K}_i} = [h_i^{k_i} \circ \mathbf{v}_i]_{k_i \in \mathcal{K}_i}^T. \quad (3.2)$$

Note that $\mathbf{f}_{\mathcal{U}_i}^{\mathbf{s-AiO}}$ corresponds to the i th bloc of vector $\mathbf{f}_{\mathcal{U}}^{\mathbf{s-AiO}}$ and is a subvector of $\mathbf{f}_{\mathcal{U}}^{\mathbf{s-AiO}}$. Therefore, from Propositions 2.3 and 3.2, we obtain the following corollary.

Corollary 3.7. Let $[\mathbf{u}_1 \cdots \mathbf{u}_p]^T$ be any vector such that $\mathbf{u}_i$ is any subvector of $\mathbf{f}_{\mathcal{U}_i}^{\mathbf{s-AiO}}$ for every $i \in [p]$. If the functions in the sets $\mathcal{H}_i$ for every $i \in [p]$ are

- (i) (strongly) increasing, then $E_{(\cdot/s)}(X, \mathbf{v}) = E_{(\cdot/s)}(X, [\mathbf{v} \ \mathbf{u}_1 \ \cdots \ \mathbf{u}_p]^T)$,
- (ii) strictly increasing, then $E_w(X, \mathbf{v}) = E_w(X, [\mathbf{v} \ \mathbf{u}_1 \ \cdots \ \mathbf{u}_p]^T)$.

Propositions 2.3 and 3.3 also yield the following result.

Corollary 3.8. Let $[\mathbf{u}_1 \cdots \mathbf{u}_p]^T$ be any vector such that $\mathbf{u}_i$ is any d_i -dimensional subvector of $\mathbf{f}_{\mathcal{U}_i}^{\mathbf{s-AiO}}$ for every $i \in [p]$. If for every $i \in [p]$, there exists an order-preserving operator $\alpha_i : \mathbb{R}^{d_i} \rightarrow \mathbb{R}^{b_i}$ such that $\alpha_i(\mathbf{u}_i) = \mathbf{v}_i + \mathbf{c}_i$ with $\mathbf{c}_i \in \mathbb{R}^{b_i}$, and if the functions in the sets $\mathcal{H}_i$ are

- (i) (strongly) increasing, then $E_{(\cdot/s)}(X, \mathbf{v}) = E_{(\cdot/s)}(X, [\mathbf{u}_1 \ \cdots \ \mathbf{u}_p]^T) = E_{(\cdot/s)}(X, [\mathbf{f}_{\mathcal{U}_1}^{\mathbf{s-AiO}} \ \cdots \ \mathbf{f}_{\mathcal{U}_p}^{\mathbf{s-AiO}}]^T)$,
- (ii) strictly increasing, then $E_w(X, \mathbf{v}) = E_w(X, [\mathbf{u}_1 \ \cdots \ \mathbf{u}_p]^T) = E_w(X, [\mathbf{f}_{\mathcal{U}_1}^{\mathbf{s-AiO}} \ \cdots \ \mathbf{f}_{\mathcal{U}_p}^{\mathbf{s-AiO}}]^T)$.

As an example of operators satisfying Corollary 3.8, for $b = 3$ and $p = 2$, we have $\mathbf{v} = [\mathbf{v}_1, \mathbf{v}_2]$, where $\mathbf{v}_1 : \mathbb{R}^n \rightarrow \mathbb{R}^2$ and $\mathbf{v}_2 = v_2 : \mathbb{R}^n \rightarrow \mathbb{R}$. For every $i \in [2]$, define $h_i \circ \mathbf{v}_i := e^{k_i} \mathbf{v}_i$, where e^{k_i} is the Euler's number raised to the k_i -th power and $k_i \in K_i = \{1, 2, \dots\}$. Then $\mathcal{U}_1 = \{ev_{11}, ev_{12}, e^2v_{11}, e^2v_{12}, \dots\}$ and $\mathcal{U}_2 = \{ev_{21}, e^2v_{21}, \dots\}$. We might consider $\mathbf{u}_1 = [ev_{11}, ev_{12}]$, $\mathbf{u}_2 = [e^2v_{21}]$, and operator α_1 defined by the matrix

$$\begin{bmatrix} e^{-1} & 0 \\ 0 & e^{-1} \end{bmatrix}$$

and operator $\alpha_2(\mathbf{v}_2) := e^{-2}\mathbf{v}_2$, where both α_1 and α_2 are order-preserving.

Similar to the vector case, we recognize the set of operators satisfying Corollary 3.8.

Definition 3.9. The sets $\mathcal{H}_1, \dots, \mathcal{H}_p$ of functions for which either

$$E_{(\cdot/s)}(X, \mathbf{v}) = E_{(\cdot/s)}\left(X, \begin{bmatrix} \mathbf{f}_{\mathcal{U}_1}^{\text{s-AiO}} & \dots & \mathbf{f}_{\mathcal{U}_p}^{\text{s-AiO}} \end{bmatrix}^T\right) \text{ or } E_w(X, \mathbf{v}) = E_w\left(X, \begin{bmatrix} \mathbf{f}_{\mathcal{U}_1}^{\text{s-AiO}} & \dots & \mathbf{f}_{\mathcal{U}_p}^{\text{s-AiO}} \end{bmatrix}^T\right)$$

are called *scalar-AiO-basis-preserving*.

Finding efficient solutions to scalar-AiO-MOP (3.1), an intractable problem due to an infinite number of objective functions, is reduced to solving a finite MOP due to Corollary 3.8.

Corollary 3.10. Let $\mathcal{U}$ be scalar-valued and for every $i \in [p]$, set $\mathcal{H}_i$ be scalar-AiO-basis-preserving. A solution $\mathbf{x} \in X$ is scalar-AiO-robust (strictly/weakly) efficient to UMOP (2.6) if and only if it is (strictly/weakly) efficient to the finite MOP (2.2).

4. ALL-AT-ONCE (AAO) APPROACH

The term AaO refers to the MDO approach in which modeling and optimization are fully integrated meaning that all of the resulting variables are optimization variables and all of the resulting equations and inequalities are optimization constraints [12, 38].

In the AaO approach we consider a conjunction of possibly infinitely many MOPs, each associated with a realization of the objective function

$$\bigwedge_{k \in \mathcal{K}} \min_{\mathbf{x} \in \mathcal{X}} \mathbf{f}^k(\mathbf{x}), \quad (4.1)$$

to define robust efficient solutions to problem (2.6). These MOPs are solved simultaneously and independently subject to the same feasible set and yield the solutions that are efficient to all these MOPs.

Definition 4.1. Let uncertainty be vector-valued. A solution $\mathbf{x} \in X$ is called *vector-AaO-robust* (strictly/weakly) efficient to UMOP (2.6) if it is (strictly/weakly) efficient to every p -objective subproblem in AaO-MOP (4.1). The set of vector-AaO-robust (strictly/weakly) efficient solutions, denoted $E_{(\cdot/s/w)}^{\text{v-AaO}}(X, \mathbf{f}, \mathcal{U})$, is defined as

$$E_{(\cdot/s/w)}^{\text{v-AaO}}(X, \mathbf{f}, \mathcal{U}) := \bigcap_{k \in \mathcal{K}} E_{(\cdot/s/w)}(X, \mathbf{f}^k),$$

where $E_{(\cdot/s/w)}(X, \mathbf{f}^k)$ denotes the (strictly/weakly) efficient set of subproblem k for every $k \in \mathcal{K}$.

This type of efficient solutions is defined for UMOLPs in [1], where the solutions are referred to as *scenario-unanimous*, and in [33], where the solutions are referred to as *highly robust efficient*. In [36], a new concept of highly efficient solutions is introduced, which, in a particular case, reduces to the highly robust efficient solutions.

We recognize that, in general, intersections of efficient sets may be empty which might be the case in the presented results. However, some presented results imply that the intersections are nonempty as long as some efficient sets remain nonempty. When the entire basis vector-valued function $\mathbf{v}$ is also considered in all the MOPs in AaO-MOP (4.1), Proposition 3.2 implies the following.

Corollary 4.2. If the functions in set $\mathcal{H}$ are

- (i) (strongly) order-preserving, then $E_{(\cdot/s)}(X, \mathbf{v}) = \bigcap_{k \in \mathcal{K}} E_{(\cdot/s)}(X, [\mathbf{v} \ \mathbf{f}^k]^T)$,
- (ii) strictly order-preserving, then $E_w(X, \mathbf{v}) = \bigcap_{k \in \mathcal{K}} E_w(X, [\mathbf{v} \ \mathbf{f}^k]^T)$.

Note that Corollary 4.2 could be stated with respect to any subset of $\mathcal{K}$ instead of the entire set $\mathcal{K}$ of indices that defines vector-valued uncertainty set $\mathcal{U}$.

Contrary to the AiO approach, drawing conditions that would allow to not explicitly include vector $\mathbf{v}$ in the MOPs in AaO-MOP (4.1) while preserving the efficient set for each realization seems unachievable. The next propositions state necessary conditions for the intersection of the (strictly/weakly) efficient sets for each realization to be included in the (strictly/weakly) efficient set to finite MOP (2.2).

Proposition 4.3. *If any one of the following two assumptions holds*

- (i) *one of the functions in $\mathcal{H}$ is order-preserving and injective,*
- (ii) *for every $j \in [b]$, there exists $k_j \in \mathcal{K}$ such that $\mathbf{h}^{k_j}$ is order-preserving, and there exist a strictly increasing function $\alpha_j : \mathbb{R} \rightarrow \mathbb{R}$ and an increasing function $\beta_j : \mathbb{R}^b \rightarrow \mathbb{R}$ such that $\alpha_j \circ v_j + \beta_j \circ \mathbf{v}$ is a component of $\mathbf{f}^{k_j}$,*

then $\bigcap_{k \in \mathcal{K}} E(X, \mathbf{f}^k) \subseteq E(X, \mathbf{v})$.

Proof. Let $\mathbf{x} \in \bigcap_{k \in \mathcal{K}} E(X, \mathbf{f}^k)$ and $\mathbf{x} \notin E(X, \mathbf{v})$. There then exists $\bar{\mathbf{x}} \in X$ such that $\mathbf{v}(\bar{\mathbf{x}}) \leq \mathbf{v}(\mathbf{x})$. If assumption (i) holds, then considering an index $k \in \mathcal{K}$ such that $\mathbf{h}^k$ is order-preserving and injective implies $\mathbf{h}^k(\mathbf{v}(\bar{\mathbf{x}})) \leq \mathbf{h}^k(\mathbf{v}(\mathbf{x}))$, that is, $\mathbf{x} \notin E(X, \mathbf{f}^k)$, a contradiction.

Without loss of generality (W.l.o.g.) suppose $v_1(\bar{\mathbf{x}}) < v_1(\mathbf{x})$. Considering $k_1 \in \mathcal{K}$, $\alpha_1 : \mathbb{R} \rightarrow \mathbb{R}$, and $\beta_1 : \mathbb{R}^b \rightarrow \mathbb{R}$ such that they satisfy the properties of assumption (ii) for $j = 1$ implies $\mathbf{h}^{k_1}(\mathbf{v}(\bar{\mathbf{x}})) \leq \mathbf{h}^{k_1}(\mathbf{v}(\mathbf{x}))$, that is, $\mathbf{x} \notin E(X, \mathbf{f}^{k_1})$, a contradiction. $\square$

As an example of operators satisfying Proposition 4.3(ii) we may consider for every $j \in [b]$, an operator $\mathbf{h}^{k_j}$ of $\mathcal{H}$ such that one of its p component functions is a quadratic function $f^{k_j} : \mathbb{R}^b \rightarrow \mathbb{R}$ defined as $f^{k_j}(\mathbf{z}) = \frac{1}{2} \mathbf{z}^T A^{k_j} \mathbf{z} + \mathbf{c}^{k_j T} \mathbf{z} + d^{k_j}$, where A^{k_j} is a $b \times b$ symmetric nonnegative matrix, $\mathbf{c}^{k_j}$ is a nonnegative vector of $\mathbb{R}^b$, $a_{jj}^{k_j} + c_j^{k_j} > 0$, and $d^{k_j} \in \mathbb{R}$. Provided $\mathbf{v}$ is a nonnegative vector-valued function on $\mathcal{X}$, $\mathbf{h}^{k_j}$ satisfies Proposition 4.3(ii) with strictly increasing function α_j defined as $\frac{1}{2} a_{jj}^{k_j} z^2 + c_j^{k_j} z + d^{k_j}$ for $z \in \mathbb{R}_{\geq}$ and increasing function β_j as $\sum_{i \in [b] \setminus \{j\}} (\frac{1}{2} a_{ii}^{k_j} z_i^2 + c_i^{k_j} z_i) + \sum_{i=1}^{b-1} \sum_{\ell=i+1}^b a_{i\ell}^{k_j} z_i z_\ell$ for $\mathbf{z} \in \mathbb{R}_{\geq}^b$. Note that a same quadratic function f^k may serve for different j in $[b]$ (e.g., for all $j \in b$ if A^k or $\mathbf{c}^k$ is positive).

Assumptions (i) and (ii) can be relaxed for strictly/weakly efficiency as stated in the next proposition.

Proposition 4.4. *If one of the functions in the set $\mathcal{H}$ is*

- (i) *order-preserving, then $\bigcap_{k \in \mathcal{K}} E_s(X, \mathbf{f}^k) \subseteq E_s(X, \mathbf{v})$,*
- (ii) *strictly order-preserving, then $\bigcap_{k \in \mathcal{K}} E_w(X, \mathbf{f}^k) \subseteq E_w(X, \mathbf{v})$,*
- (iii) *strongly order-preserving, then $\bigcap_{k \in \mathcal{K}} E_w(X, \mathbf{f}^k) \subseteq E(X, \mathbf{v})$.*

Proof. (i) Let $\mathbf{x} \in \bigcap_{k \in \mathcal{K}} E_s(X, \mathbf{f}^k)$ and $\mathbf{x} \notin E_s(X, \mathbf{v})$. There then exists $\bar{\mathbf{x}} \in X$ such that $\bar{\mathbf{x}} \neq \mathbf{x}$

and $\mathbf{v}(\bar{\mathbf{x}}) \leq \mathbf{v}(\mathbf{x})$. For every $k \in \mathcal{K}$ such that $\mathbf{h}^k$ is order-preserving, we have $\mathbf{h}^k(\mathbf{v}(\bar{\mathbf{x}})) \leq \mathbf{h}^k(\mathbf{v}(\mathbf{x}))$, that is, $\mathbf{x} \notin E_s(X, \mathbf{f}^k)$, a contradiction.

(ii) The proof for weakly efficient solutions is analogous to the one of (i).

(iii) Let $\mathbf{x} \in \bigcap_{k \in \mathcal{K}} E_w(X, \mathbf{f}^k)$ and $\mathbf{x} \notin E(X, \mathbf{v})$. There then exists $\bar{\mathbf{x}} \in X$ such that $\mathbf{v}(\bar{\mathbf{x}}) \leq \mathbf{v}(\mathbf{x})$.

Consider an index $k \in \mathcal{K}$ such that $\mathbf{h}^k$ is strongly order-preserving. We then obtain $\mathbf{h}^k(\mathbf{v}(\bar{\mathbf{x}})) < \mathbf{h}^k(\mathbf{v}(\mathbf{x}))$, that is, $\mathbf{x} \notin E_w(X, \mathbf{f}^k)$, a contradiction. $\square$

We complete this section with two corollaries resulting from the presented work.

Corollary 4.5. *Let $\mathcal{U}$ be vector-valued and the assumptions of Propositions 4.3 and 4.4 hold. If a solution $\mathbf{x} \in X$ is vector-AaO-robust (strictly/weakly) efficient to UMOP (2.6) then it is (strictly/weakly) efficient to the finite MOP (2.2).*

Corollary 4.6. *Let the assumptions of Corollaries 3.5 and 4.5 hold. Then*

$$E_{(\cdot/s/w)}^{v\text{-AaO}}(X, \mathbf{f}, \mathcal{U}) \subseteq E_{(\cdot/s/w)}(X, \mathbf{f}_{\mathcal{U}}^{v\text{-AiO}}).$$

4.1. Scalar-valued uncertainty. For the scalar-valued uncertainty, AaO-MOP (4.1) can be written as a possibly infinite conjunction of MOPs

$$\bigwedge_{(k_1, \dots, k_p) \in \tilde{\mathcal{K}}} \min_{\mathbf{x} \in \mathcal{X}} \begin{bmatrix} f_1^{k_1}(\mathbf{x}) & \dots & f_p^{k_p}(\mathbf{x}) \end{bmatrix}^T, \quad (4.2)$$

where $\tilde{\mathcal{K}} := \mathcal{K}_1 \times \dots \times \mathcal{K}_p$.

Definition 4.7. Let uncertainty be scalar-valued. A solution $\mathbf{x} \in X$ is called *scalar-AaO-robust (strictly/weakly) efficient* to UMOP (2.6) if it is (strictly/weakly) efficient to every p -objective subproblem in (4.2). The set of scalar-AaO-robust (strictly/weakly) efficient solutions, denoted as $E_{(\cdot/s/w)}^{s\text{-AaO}}(X, \mathbf{f}, \mathcal{U})$, is defined as

$$E_{(\cdot/s/w)}^{s\text{-AaO}}(X, \mathbf{f}, \mathcal{U}) := \bigcap_{\tilde{k} \in \tilde{\mathcal{K}}} E_{(\cdot/s/w)}(X, \mathbf{f}^{\tilde{k}}),$$

where $E_{(\cdot/s/w)}(X, \mathbf{f}^{\tilde{k}})$ denotes the (strictly/weakly) efficient set of subproblem $\tilde{k}$ for all $\tilde{k} \in \tilde{\mathcal{K}}$.

The scalar-valued uncertainty and the AaO approach are first proposed by Bitran [6] in the context of interval multiobjective linear programs (MOLPs) whose solutions are named *necessarily efficient*. Methods for finding such solutions are presented by [4, 45, 42, 37]. In robust multiobjective optimization, this model is also studied for problems with uncertain parameters in the objective functions and the concept of highly robust efficient solutions is applied. Kuroiwa and Lee [54] propose the objective-wise worst-case MOP, whose efficient solutions are called robust for the original UMOP, scalarization methods for computing these solutions, and robust optimality conditions. Ehrgott et al. [18] introduce the concept of objective-wise uncertainty that allows every objective function to have its own uncertainty set and prove that in this case polytopal uncertainty sets can be reduced to discrete uncertainty sets. For UMOLPs, Goberna et al. [34] continue their work on the radius of highly robust efficiency, while [15, 16] characterize highly robust efficient solutions using the cone of improving directions, derive a counterpart MOLP, and propose computational methods.

Since the scalar-valued uncertainty model is a special case of the vector-valued uncertainty, the results in this section are presented without proofs. We start with conditions that make the efficient set of finite MOP (2.2) equal to the intersection of the efficient sets of AaO-MOP (4.2).

Corollary 4.8. *If the functions in the sets $\mathcal{H}_i$ for every $i \in [p]$ are*

- (i) *(strongly) increasing, then $E_{(\cdot/s)}(X, \mathbf{v}) = \bigcap_{\tilde{k} \in \tilde{\mathcal{K}}} E_{(\cdot/s)}\left(X, \begin{bmatrix} \mathbf{v} & \mathbf{f}^{\tilde{k}} \end{bmatrix}^T\right)$,*
- (ii) *strictly increasing, then $E_w(X, \mathbf{v}) = \bigcap_{\tilde{k} \in \tilde{\mathcal{K}}} E_w\left(X, \begin{bmatrix} \mathbf{v} & \mathbf{f}^{\tilde{k}} \end{bmatrix}^T\right)$.*

Note that Corollary 4.8 could be stated with respect to any subset of $\tilde{\mathcal{K}}$.

As we did in the AaO vector approach, we specify the necessary conditions for the intersection of the (strictly/weakly) efficient sets for each realizations to be included in the (strictly/weakly) efficient set of (2.2).

Corollary 4.9. *If for every $i \in [p]$, any one of the following two assumptions holds*

(i) one of the functions in $\mathcal{H}_i$ is either increasing and injective or strongly increasing,
(ii) for every $j \in [b_i]$, there exist $k_{ij} \in \mathcal{K}_i$, a strictly increasing function $\alpha_{ij} : \mathbb{R} \rightarrow \mathbb{R}$, and an increasing function $\beta_{ij} : \mathbb{R}^{b_i} \rightarrow \mathbb{R}$ such that $h_i^{k_{ij}} \circ \mathbf{v}_i = \alpha_{ij} \circ v_{ij} + \beta_{ij} \circ \mathbf{v}_i$,
then $\bigcap_{\tilde{k} \in \tilde{\mathcal{K}}} E(X, \mathbf{f}^{\tilde{k}}) \subseteq E(X, \mathbf{v})$.

As an example of operators satisfying Corollary 4.9(ii) we may consider for every $i \in [p]$ and every $j \in [b_i]$, an operator $h_i^{k_{ij}} = \frac{1}{2} \mathbf{z}_i^T A^{k_{ij}} \mathbf{z}_i + \mathbf{c}^{k_{ij}} \mathbf{z}_i + d^{k_{ij}}$, where $A^{k_{ij}}$ is a $b_i \times b_i$ symmetric nonnegative matrix, $\mathbf{c}^{k_{ij}}$ is a nonnegative vector of $\mathbb{R}^{b_i}$, $a_{jj}^{k_{ij}} + c_j^{k_{ij}} > 0$, and $d^{k_{ij}} \in \mathbb{R}$. Provided $\mathbf{v}_i$ is a nonnegative vector-valued function on $\mathcal{X}$, $\mathbf{h}^{k_{ij}}$ satisfies Corollary 4.9(ii) with strictly increasing function α_{ij} defined as $\frac{1}{2} a_{jj}^{k_{ij}} z^2 + c_j^{k_{ij}} z + d^{k_{ij}}$ for $z \in \mathbb{R}_{\geq}$ and increasing function β_{ij} as $\sum_{\ell \in [b] \setminus \{j\}} (\frac{1}{2} a_{\ell\ell}^{k_{ij}} z_{\ell}^2 + c_{\ell}^{k_{ij}} z_{\ell}) + \sum_{\ell=1}^{b_i-1} \sum_{r=\ell+1}^{b_i} a_{\ell r}^{k_{ij}} z_{\ell} z_r$ for $\mathbf{z}_i \in \mathbb{R}_{\geq}^{b_i}$. Note that a same quadratic function f^{k_i} may served for different j in $[b_i]$ (e.g., for every $j \in [b_i]$ if A^{k_i} or $\mathbf{c}^{k_i}$ is positive).

Corollary 4.10. *If for every $i \in [p]$, one of the functions in the set $\mathcal{H}_i$ is*

- (i) increasing, then $\bigcap_{\tilde{k} \in \tilde{\mathcal{K}}} E_s(X, \mathbf{f}^{\tilde{k}}) \subseteq E_s(X, \mathbf{v})$,
(ii) either increasing and injective or strictly increasing, then $\bigcap_{\tilde{k} \in \tilde{\mathcal{K}}} E_w(X, \mathbf{f}^{\tilde{k}}) \subseteq E_w(X, \mathbf{v})$.

Based on Corollaries 4.9 and 4.10, conjunction AaO-MOP (4.1) is related to MOP (2.2) as follows.

Corollary 4.11. *Let $\mathcal{U}$ be scalar-valued and the assumptions of Corollaries 4.9 and 4.10 hold. If a solution $\mathbf{x} \in X$ is scalar-AaO-robust (strictly/weakly) efficient to UMOP (2.6) then it is (strictly/weakly) efficient to finite MOPs (2.2).*

4.2. Outer approximation of the scalar-AaO-robust efficient set. We recognize that the assumption of injectivity of the operators required in Corollaries 4.9 and 4.10 is difficult to check and therefore propose a different approach to deal with conjunction (4.2). In this approach we build a tighter outer approximation of the scalar-AaO-robust efficient set than that provided by the efficient set of problem (2.2).

In order to formulate another finite conjunction of MOPs we use the real-valued functions in the p sets

$$\mathcal{G}_i := \{v_{i1} + g_{i1} \circ \mathbf{v}_i, v_{i2} + g_{i2} \circ \mathbf{v}_i, \dots, v_{ib_i} + g_{ib_i} \circ \mathbf{v}_i\}, \quad (4.3)$$

where $g_{ij} : \mathbb{R}^{b_i} \rightarrow \mathbb{R}$ is a given function for every $i \in [p]$ and every $j \in [b_i]$, and construct the set of all p -tuples such that the i -th component of a p -tuple is a function in $\mathcal{G}_i$ for every $i \in [p]$. The number of all such p -tuples is $d := \prod_{i=1}^p \binom{b_i}{1} = \prod_{i=1}^p b_i$. The p -tuples yield vector-valued functions $\mathbf{w}_{\ell}$, for every $\ell \in [d]$, which make up the set $\mathcal{W} := \{\mathbf{w}_1, \dots, \mathbf{w}_d\}$.

Using the vector-valued functions $\mathbf{w}_{\ell}$ for every $\ell \in [d]$, where $\mathbf{w}_{\ell i}$ denote their i -th component for every $i \in [p]$, we define a finite MOP conjunction with d subproblems and each having p -objective functions

$$\bigwedge_{\ell \in [d]} \min_{\mathbf{x} \in X} [w_{\ell 1}(\mathbf{x}) \quad \dots \quad w_{\ell p}(\mathbf{x})]^T. \quad (4.4)$$

Definition 4.12. A solution $\mathbf{x} \in X$ is called (strictly/weakly) efficient to MOP conjunction (4.4) if it is (strictly/weakly) efficient to every p -objective subproblem in (4.4). The set of (strictly/weakly) efficient

solutions to (4.4), denoted as $E_{(\cdot/s/w)}^{s\text{-AaO}}(X, \mathcal{W})$, is defined as

$$E_{(\cdot/s/w)}^{s\text{-AaO}}(X, \mathcal{W}) := \bigcap_{\mathbf{w}_\ell \in \mathcal{W}} E_{(\cdot/s/w)}(X, \mathbf{w}_\ell),$$

where for every $\ell \in [d]$, $E_{(\cdot/s/w)}(X, \mathbf{w}_\ell)$ denotes the (strictly/weakly) efficient set of subproblem.

We show the inclusion of the efficient set of finite conjunction (4.4) in the efficient set of finite MOP (2.2).

Proposition 4.13. *If for every $i \in [p]$ and every $j \in [b_i]$ function g_{ij} is increasing then $E_{(\cdot/s/w)}^{s\text{-AaO}}(X, \mathcal{W}) \subseteq E_{(\cdot/s/w)}(X, \mathbf{v})$.*

Proof. We prove the result for efficient solutions. Let $\mathbf{x} \in E^{s\text{-AaO}}(X, \mathcal{W}) = \bigcap_{\mathbf{w}_\ell \in \mathcal{W}} E(X, \mathbf{w}_\ell)$, that is, $\mathbf{x} \in E(X, \mathbf{w}_\ell)$ for every $\ell \in [d]$. Suppose that $\mathbf{x} \notin E(X, \mathbf{v})$. Thus, there exists an $\bar{\mathbf{x}} \in X$ such that $\mathbf{v}(\bar{\mathbf{x}}) \leq \mathbf{v}(\mathbf{x})$, which implies that there exists at least one $i \in [p]$ such that $\mathbf{v}_i(\bar{\mathbf{x}}) \leq \mathbf{v}_i(\mathbf{x})$, or equivalently, there exists at least one $j \in [b_i]$ such that $v_{ij}(\bar{\mathbf{x}}) < v_{ij}(\mathbf{x})$. W.l.o.g., let $i = j = 1$. Therefore, we have

$$\begin{aligned} \mathbf{v}_1(\bar{\mathbf{x}}) &\leq \mathbf{v}_1(\mathbf{x}), \\ [\mathbf{v}_2(\bar{\mathbf{x}}) \quad \cdots \quad \mathbf{v}_p(\bar{\mathbf{x}})]^T &\leq [\mathbf{v}_2(\mathbf{x}) \quad \cdots \quad \mathbf{v}_p(\mathbf{x})]^T. \end{aligned}$$

Construct a p -tuple such that $v_{11} + g_{11} \circ \mathbf{v}_1$ is its first component. This p -tuple is not unique, as the remaining $p - 1$ components may be selected in any combination. Consider w.l.o.g. the p -tuple $\mathbf{w}_\ell = [(v_{11} + g_{11} \circ \mathbf{v}_1) \quad w_{\ell 2} \quad \cdots \quad w_{\ell p}]^T$ for some $\ell \in [d]$. Therefore, because $v_{11}(\bar{\mathbf{x}}) + g_{11}(\mathbf{v}_1(\bar{\mathbf{x}})) < v_{11}(\mathbf{x}) + g_{11}(\mathbf{v}_1(\mathbf{x}))$ and $w_{\ell i}(\bar{\mathbf{x}}) \leq w_{\ell i}(\mathbf{x})$ for every $i \in \{2, \dots, p\}$, it must hold $\mathbf{w}_\ell(\bar{\mathbf{x}}) \leq \mathbf{w}_\ell(\mathbf{x})$, and $\mathbf{x} \notin E(X, \mathbf{w}_\ell)$ for some $\ell \in [d]$, a contradiction. The proofs for strictly and weakly efficient solutions are analogous. $\square$

To relate the finite conjunction (4.4) to the infinite conjunction (4.2) we have to make sure that the sets $\mathcal{H}_i$, for every $i \in [p]$, contain appropriate operators. Assume that for every $i \in [p]$, set $\mathcal{H}_i$ contains a function $h_i^{k_i(j_i)}$ such that $h_i^{k_i(j_i)} \circ \mathbf{v}_i$ is the image of function $v_{ij_i} + g_{ij_i} \circ \mathbf{v}_i$ of $\mathcal{G}_i$ by some strictly-increasing function for some $k_i(j_i) \in \mathcal{K}_i$ and $j_i \in [b_i]$. Under this assumption among all scalar realizations indexed by $\tilde{k} \in \tilde{\mathcal{K}}$ (see the definition of conjunction (4.2)) there are realizations that assume the form of p -tuples $\mathbf{w}_\ell, \ell \in [d]$, making up the set $\mathcal{W}$. We now prove that the efficient set of the finite conjunction (4.4) provides a bound set for the efficient set of the infinite conjunction (4.2). The assumption under which Proposition 4.14 holds is similar to assumption (ii) in Corollary 4.9.

Proposition 4.14. *Let $\mathcal{U}$ be scalar-valued and assume that for every $i \in [p]$ and every $j_i \in [b_i]$, there exist $h_i^{k_i(j_i)} \in \mathcal{H}_i$ and a strictly increasing function $\alpha_{ij_i} : \mathbb{R} \rightarrow \mathbb{R}$ such that $h_i^{k_i(j_i)} \circ \mathbf{v}_i = \alpha_{ij_i} \circ (v_{ij_i} + g_{ij_i} \circ \mathbf{v}_i)$. If a solution $\mathbf{x} \in X$ is scalar-AaO-robust (strictly/weakly) efficient to UMOP (2.6) then it is (strictly/weakly) efficient to every p -objective subproblem in the MOP conjunction (4.4).*

Proof. We prove the result for efficient solutions. Suppose $\mathbf{x} \in \bigcap_{\tilde{k} \in \tilde{\mathcal{K}}} E(X, \mathbf{f}^{\tilde{k}})$ and $\mathbf{x} \notin \bigcap_{\mathbf{w}_\ell \in \mathcal{W}} E(X, \mathbf{w}_\ell)$.

There then exist $\bar{\ell} \in [d]$ and $\bar{\mathbf{x}} \in X$ such that $\mathbf{w}_{\bar{\ell}}(\bar{\mathbf{x}}) \leq \mathbf{w}_{\bar{\ell}}(\mathbf{x})$. From the definition of p -tuple $\mathbf{w}_{\bar{\ell}}$, for every $i \in [p]$, component $w_{\bar{\ell}i}$ corresponds to a function $(v_{ij_i} + g_{ij_i} \circ \mathbf{v}_i)$, $j_i \in [b_i]$, in $\mathcal{G}_i$. The existence of $h_i^{k_i(j_i)}$ and α_{ij_i} for every $i \in [p]$ and every $j_i \in [b_i]$ implies that there exists $\bar{k} := (k_{i_{j_1}}, k_{i_{j_2}}, \dots, k_{i_{j_p}}) \in \tilde{\mathcal{K}}$ such that

$$\mathbf{f}^{\bar{k}} = \mathbf{h}^{\bar{k}} \circ \mathbf{v} = \begin{bmatrix} \alpha_{1j_1} \circ (v_{1j_1} + g_{1j_1} \circ \mathbf{v}_1) \\ \alpha_{2j_2} \circ (v_{2j_2} + g_{2j_2} \circ \mathbf{v}_2) \\ \vdots \\ \alpha_{pj_p} \circ (v_{pj_p} + g_{pj_p} \circ \mathbf{v}_p) \end{bmatrix}.$$

From $\alpha_{1j_1}, \dots, \alpha_{pj_p}$ being strictly increasing and $\mathbf{w}_{\bar{\ell}}(\bar{\mathbf{x}}) \leq \mathbf{w}_{\bar{\ell}}(\mathbf{x})$, we obtain $\mathbf{f}^{\bar{k}}(\bar{\mathbf{x}}) \leq \mathbf{f}^{\bar{k}}(\mathbf{x})$, a contradiction to $\mathbf{x} \in \bigcap_{\bar{k} \in \bar{\mathcal{K}}} E(X, \mathbf{f}^{\bar{k}})$. The proofs for strictly and weakly efficient solutions are analogous. $\square$

The intersection of the efficient sets of the MOP conjunction (4.4) may be useful when approximating the scalar-AaO-robust efficient sets of UMOP (2.6).

Corollary 4.15. *Let $\mathcal{U}$ be scalar-valued and the assumptions of Propositions 4.13 and 4.14 hold. Then*

$$E_{(\cdot/s/w)}^{s\text{-AaO}}(X, \mathbf{f}, \mathcal{U}) \subseteq E_{(\cdot/s/w)}^{s\text{-AaO}}(X, \mathcal{W}) \subseteq E_{(\cdot/s/w)}(X, \mathbf{v}).$$

In comparison to Corollaries 4.9 and 4.10, Corollary 4.15 offers the same result under a rather mild assumption that may not be satisfied in every application but if it is, it eliminates the need to check the monotonicity and injectivity of the operators. Another containment result is straightforward.

Corollary 4.16. *Let the assumptions of Corollaries 3.10 and 4.15 hold. Then*

$$E_{(\cdot/s/w)}^{s\text{-AaO}}(X, \mathbf{f}, \mathcal{U}) \subseteq E_{(\cdot/s/w)}(X, \mathbf{f}_{\mathcal{U}}^{s\text{-AiO}}).$$

5. DISCUSSION

The AiO and AaO approaches to handling surrogate UMOP (2.6) can be compared to each other and to other concepts of resolving uncertainty. In the first comparison we use two important perspectives: the preservation of the robust efficient set and the modeling of uncertainty.

The two approaches differ with respect to the preservation. The AiO-robust efficient set is preserved by the efficient set to (2.2) as long as the operators defining the uncertainty sets satisfy certain properties. The AaO-robust efficient set is not preserved but outer-approximated by the efficient set to MOP conjunction (4.4) and the efficient set to (2.2).

The AiO and AaO approaches model uncertainty in two very different ways. In both models, all realizations of all uncertain objective functions are assumed to happen simultaneously, which is unlikely or might be even impossible, while the resulting efficient sets are computed differently. If the conditions for preserving efficiency do not hold, the AiO-robust efficient solutions exhibit weakness resulting from the fact that if an objective function is dropped for the multiobjective model, an efficient solution may not remain efficient but become ϵ -efficient [27]. Therefore an AiO-robust efficient solutions may lose efficiency and be ϵ -efficient for a subset of uncertainty realizations or a particular realization. The ϵ is a measure of the objective-space distance of that ϵ -efficient solution to the efficient set and can be computed using Benson's method [3].

AiO-robust efficient solutions can be compared to flimsily robust-efficient solutions defined as the solutions that are efficient for at least one scenario realization [43]. This comparison is in favor of the latter due to the fact that an efficient solution remains weakly efficient when an objective function is added to the multiobjective model. Consequently, flimsily robust-efficient solutions become weakly efficient for two or more uncertainty realizations.

In contrast, every AaO-robust efficient solution remains efficient for each uncertainty realization, which makes this concept strong. On the other hand, the AaO model is also more demanding since it relies on the intersection of the efficient sets that shall be nonempty. As long as this is the case and because the AaO-robust efficient solutions are also AiO-robust efficient, the former are likely to be the more preferred solutions for the decision maker. However, if the AaO-robust efficient set is empty, which might be likely in some applications, the AiO-robust efficient set being always nonempty comes as a safe option.

We have seen that for both AiO and AaO approaches, the scalar models are special cases of the vector models. For $p = 1$ the two models coincide and one may think of the operators in the form of real-valued scalarizing functions. Despite this connection, the scalar case has modeling flexibility that the vector case lacks. The scalar case allows to use two types of modeling, integration or partitioning.

The creation of the tuples, which convert the AaO infinite conjunction into a finite one, is an example of integration which provides an efficient set “filling the gap” between the efficient set of (2.2) and the scalar-AaO-robust efficient set. Handling each group independently of the others allows partitioning of the criteria. For example, groups of multiple objectives can be scalarized independently of each other because of their different decision-making context, as exercised in [46]. Alternatively, the groups can be modeled independently of each other, or every objective function may be associated with its own uncertainty set as exploited in the concept of the objective-wise uncertainty.

Additionally, for $p \gg 2$, the partitioning of the objective functions in (2.2) into subvectors seems analogous to the decomposition approach of a finite MOP with a large number of criteria into a collection of multiobjective subproblems with each heaving a smaller number of criteria [27, 74]. In spite of this resemblance, these studies have reverse motivation. The current interest is to examine the growing number of criteria, while the past focus is to avoid solving the original MOP but construct its efficient solutions by solving only the subproblems.

The preservation or approximation of the AiO-robust efficient set offers a new perspective on linear and monotone systems performing under multiple objectives and function uncertainty. Since these systems inherently possess the properties to preserve or approximate this set for any collection of basis functions, there is no need to construct an uncertainty set while its existence is given implicitly through these systems’ performance. If a system is nonlinear or non-monotone, it may be considered linear or monotone in a small range of its performance and the properties will apply. In any case, the choice of representative scenarios is a fundamental decision and strongly affects the AiO-robust efficient set. On the other hand, this set is computable as long as the efficient set to (2.2) is, which is very likely given the advanced state-of-the-art of computational multiobjective optimization ([19]). This makes UMOPs under function uncertainty competitive with UMOPs under parameter uncertainty that still lack algorithms for the computation of their robust efficient solutions.

To make the paper complete mathematically and show a variety of operators, in the next section we build uncertainty sets allowing for the preservation of AiO-robust efficient sets.

6. LINEAR OPERATORS

To represent the uncertainty sets for the vector- and scalar-valued uncertainty cases, we consider linear operators represented by real matrices that construct functions making up these sets. Despite the fact that linear operators are well established, we delve into their properties since these operators can model the performance of numerous systems in engineering and science that obey the superposition principle. We first briefly review the use of linear operators in multiobjective optimization and then present the conditions for constructing basis preserving operators so that the resulting uncertainty sets preserve efficiency. We include examples of specific linear operators that satisfy these conditions. Notice that the results presented in this section can be straightforwardly extended to affine operators.

Let $\mathcal{H} := \{\mathbf{h} : \mathbb{R}^b \rightarrow \mathbb{R}^p, \mathbf{h}(\mathbf{z}) = M\mathbf{z}, M \in \mathcal{M}\}$ be the set of linear operators associated with a specific set $\mathcal{M}$ of $p \times b$ real matrices. Then, the vector-valued uncertainty set $\mathcal{U}$ assumes the form

$$\mathcal{U} := \{\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^p, \mathbf{f} = M\mathbf{v}, M \in \mathcal{M}\} = \{\mathbf{f}^k : \mathbb{R}^n \rightarrow \mathbb{R}^p, \mathbf{f}^k = M^k\mathbf{v}, k \in \mathcal{K}\}. \quad (6.1)$$

Based on [63], the modification of the basis functions by a matrix can be interpreted as solving the finite MOP (2.2) with respect to the cone defined by the matrix M rather than with respect to the standard definition of Pareto-efficiency whose cone is defined with an identity matrix. In this case, every realization of the UMOP reflects different preferences carried by the cone and applied to MOP (2.2) which may originate from multiple decision makers engaged in the decision process. We refer the reader to [40] for the theory of this kind of preference modeling and to [39, 7] for its application in automotive design. In this case, as long as the objective functions in (2.2) are affine, the function uncertainty reduces to parameter uncertainty with uncertain objective function coefficients.

In a more general approach, the elements of matrix M are defined to be functions of an uncertain input to model uncertain tradeoffs between criteria and to lead to an uncertain preference cone [75]. In this case, as long as the objective functions in (2.2) and the functions being the elements of M are affine, the same reduction takes place.

To develop conditions leading to preserving efficiency, we narrow down the set $\mathcal{H}$ into $\mathcal{H}^\geq := \{\mathbf{h} : \mathbb{R}^b \rightarrow \mathbb{R}^p, \mathbf{h}(\mathbf{z}) = M\mathbf{z}, M \in \mathcal{M}^\geq\}$, where $\mathcal{M}^\geq \subseteq \mathcal{M}$ is the set of $p \times b$ nonnegative (and nonzero) real matrices. Every function in $\mathcal{H}^\geq$ may be viewed as a way to obtain p conical combinations of the basis function $v_1, \dots, v_b$. For every subset $\mathcal{S} \subseteq \mathcal{M}^\geq$, let $\mathcal{H}^\mathcal{S} := \{\mathbf{h} : \mathbb{R}^b \rightarrow \mathbb{R}^p, \mathbf{h}(\mathbf{z}) = M\mathbf{z}, M \in \mathcal{S}\}$ denote the set of operators whose matrices belong to $\mathcal{S}$, and $\mathcal{U}^\mathcal{S}$ denote the associated vector-valued uncertainty set,

$$\mathcal{U}^\mathcal{S} := \{\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^p, \mathbf{f} = M\mathbf{v}, M \in \mathcal{S}\}. \quad (6.2)$$

The next result presents a sufficient condition for $\mathcal{H}^\mathcal{S}$ to be vector-AiO-basis-preserving.

Proposition 6.1. *Let $\mathcal{S} \subseteq \mathcal{M}^\geq$. If for every $j \in [b]$ there exists a matrix M in $\mathcal{S}$ having one of its rows being a positive multiple of the j th unit vector $\mathbf{e}_j^T \in \mathbb{R}^b$, then the set $\mathcal{H}^\mathcal{S}$ is vector-AiO-basis-preserving.*

Proof. Under the stated assumption, there exists a $b \times b$ diagonal matrix P with positive diagonal elements such that $P\mathbf{v}$ is a subvector of $\mathbf{f}_{\mathcal{U}^\mathcal{S}}^{\mathbf{v-AiO}}$. The linear transformation associated with P^{-1} fulfills the assumptions of Proposition 3.3. As all the matrices defining the functions of $\mathcal{H}^\mathcal{S}$ are nonnegative, all these functions are order-preserving. Consequently, Proposition 3.3 yields that $\mathcal{H}^\mathcal{S}$ is vector-AiO-basis-preserving. $\square$

Note that for the case of linear operators, vector $\mathbf{u}$ of $\mathbf{f}_{\mathcal{U}}^{\mathbf{v-AiO}}$, considered in Proposition 3.3, corresponds to the image of $\mathbf{v}$ by a linear transformation defined by a $d \times b$ matrix P . A nonnegative left-inverse matrix of P would thus be sufficient to define the order-preserving (and then injective) operator α in Proposition 3.3.

Not all the sets $\mathcal{S}$, associated with the operator sets $\mathcal{H}^\mathcal{S}$, satisfy the assumption of Proposition 6.1. It is the case, for instance, when for all the matrices in $\mathcal{S}$, at least two entries per row take positive values. The next proposition states a necessary condition for the efficient solution with respect to $\mathbf{f}_{\mathcal{U}^\mathcal{S}}^{\mathbf{v-AiO}}$ to be efficient with respect to $\mathbf{v}$.

Proposition 6.2. *Let $\mathcal{S} \subseteq \mathcal{M}^\geq$. If for every $j \in [b]$ there exists a matrix in $\mathcal{S}$ with nonzero j^{th} column then $E(X, \mathbf{f}_{\mathcal{U}^\mathcal{S}}^{\mathbf{v-AiO}}) \subseteq E(X, \mathbf{v})$.*

Proof. Let $\mathbf{x} \in E(X, \mathbf{f}_{\mathcal{U}^\mathcal{S}}^{\mathbf{v-AiO}})$ and $\mathbf{x} \notin E(X, \mathbf{v})$. There then exists $\bar{\mathbf{x}} \in X$ such that $\mathbf{v}(\bar{\mathbf{x}}) \leq \mathbf{v}(\mathbf{x})$. Suppose w.l.o.g. that $v_1(\bar{\mathbf{x}}) < v_1(\mathbf{x})$. For all the matrices in $\mathcal{S}$ are nonnegative, we obtain $M\mathbf{v}(\bar{\mathbf{x}}) \leq M\mathbf{v}(\mathbf{x})$ for every $M \in \mathcal{S}$. Consider any matrix $M^+ \in \mathcal{S}$ having a nonzero first column. For every $i \in [p]$ such that $M_{i,1}^+ > 0$, we have $(M^+\mathbf{v}(\bar{\mathbf{x}}))_i < (M^+\mathbf{v}(\mathbf{x}))_i$ and therefore, $M^+\mathbf{v}(\bar{\mathbf{x}}) \leq M^+\mathbf{v}(\mathbf{x})$. Consequently $\mathbf{f}_{\mathcal{U}^\mathcal{S}}^{\mathbf{v-AiO}}(\bar{\mathbf{x}}) \leq \mathbf{f}_{\mathcal{U}^\mathcal{S}}^{\mathbf{v-AiO}}(\mathbf{x})$, a contradiction to $\mathbf{x} \in E(X, \mathbf{f}_{\mathcal{U}^\mathcal{S}}^{\mathbf{v-AiO}})$. $\square$

We obtain a similar result for strictly and weakly efficient solutions as stated in the next proposition.

Proposition 6.3. *Let $\mathcal{S} \subseteq \mathcal{M}^\geq$. We have*

- (i) $E_s(X, \mathbf{f}_{\mathcal{U}^\mathcal{S}}^{\mathbf{v-AiO}}) \subseteq E_s(X, \mathbf{v})$ and
- (ii) if no matrix in $\mathcal{S}$ has an all-zero row, $E_w(X, \mathbf{f}_{\mathcal{U}^\mathcal{S}}^{\mathbf{v-AiO}}) \subseteq E_w(X, \mathbf{v})$.

Proof. The proofs for both cases are analogous to the one of Proposition 6.2. $\square$

For the reverse containment, additional assumptions on sets $\mathcal{S}$ need to be introduced. Let A be composed of all the integers j of $[b]$ for which there exists a matrix in $\mathcal{S}$ having one of its rows being a positive multiple of $\mathbf{e}_j^T$ and let $B = [b] \setminus A$. Notice that A is composed of the subscripts of the basis functions in $\mathbf{v}$ that appear (up to positive multiple scalars) in $\mathbf{f}_{\mathcal{U}^\mathcal{S}}^{\mathbf{v-AiO}}$.

Proposition 6.4. *Let $\mathcal{S} \subseteq \mathcal{M}^\geq$ be an infinite set and $\{A, B\}$ be a partition of $[b]$ as described above such that $A \neq \emptyset$. If any one of the following two assumptions holds*

- (i) $B = \emptyset$,
- (ii) *for every $\bar{j} \in B$, there exists a sequence $(M^{\bar{j},1}, M^{\bar{j},2}, \dots)$ of matrices in $\mathcal{S}$ such that*
 - (1) *for every matrix $M^{\bar{j},q}$ in this sequence, there exists $i_q \in [p]$ such that the entries in the i_q^{th} row of this matrix satisfy*
 - (a) $M_{i_q, \bar{j}}^{\bar{j},q} < \infty \quad \forall j \in A$
 - (b) $M_{i_q, \bar{j}}^{\bar{j},q} = 0 \quad \forall j \in B \setminus \{\bar{j}\}$
 - (c) $M_{i_q, \bar{j}}^{\bar{j},q} > 0$
 - (2) $\lim_{q \rightarrow \infty} M_{i_q, \bar{j}}^{\bar{j},q} = \infty$

then $E_{(\cdot/s/w)}(X, \mathbf{v}) \subseteq E_{(\cdot/s/w)}(X, \mathbf{f}_{\mathcal{U}^S}^{\mathbf{v}\text{-AiO}})$.

Proof. We prove the result for efficient solutions. For $B = \emptyset$, the containment is implied by Proposition 3.3.(i). Suppose $B \neq \emptyset$. Let $\mathbf{x} \in E(X, \mathbf{v})$ and $\mathbf{x} \notin E(X, \mathbf{f}_{\mathcal{U}^S}^{\mathbf{v}\text{-AiO}})$. Then, there exists $\bar{\mathbf{x}} \in X$ such that

$$\mathbf{f}_{\mathcal{U}^S}^{\mathbf{v}\text{-AiO}}(\bar{\mathbf{x}}) \leq \mathbf{f}_{\mathcal{U}^S}^{\mathbf{v}\text{-AiO}}(\mathbf{x}). \quad (6.3)$$

Let $\bar{j} \in B$. Consider a matrix $M^{\bar{j},q}$ in the sequence associated with $\bar{j}$. For all the components of vector $M^{\bar{j},q}$ also appear in vector $\mathbf{f}_{\mathcal{U}^S}^{\mathbf{v}\text{-AiO}}$, we have by (6.3)

$$\left(\sum_{j \in A} M_{i_q, j}^{\bar{j},q} v_j(\bar{\mathbf{x}}) \right) + M_{i_q, \bar{j}}^{\bar{j},q} v_{\bar{j}}(\bar{\mathbf{x}}) \leq \left(\sum_{j \in A} M_{i_q, j}^{\bar{j},q} v_j(\mathbf{x}) \right) + M_{i_q, \bar{j}}^{\bar{j},q} v_{\bar{j}}(\mathbf{x}),$$

that is,

$$v_{\bar{j}}(\mathbf{x}) - v_{\bar{j}}(\bar{\mathbf{x}}) \geq \frac{\sum_{j \in A} M_{i_q, j}^{\bar{j},q} (v_j(\bar{\mathbf{x}}) - v_j(\mathbf{x}))}{M_{i_q, \bar{j}}^{\bar{j},q}}. \quad (6.4)$$

Consider a basis function v_j with $j \in A$ and let M^j be a matrix in $\mathcal{S}$ with a row being a positive multiple of vector $\mathbf{e}_j^T$. W.l.o.g. suppose that the first row of M^j is $\alpha \mathbf{e}_i^T$ where $\alpha > 0$. The first component of vector $M\mathbf{v}$ is therefore αv_j . Consequently, αv_j is also a component of $\mathbf{f}_{\mathcal{U}^S}^{\mathbf{v}\text{-AiO}}$. From (6.3), we thus deduce $v_j(\bar{\mathbf{x}}) \leq v_j(\mathbf{x})$. Hence

$$v_j(\bar{\mathbf{x}}) \leq v_j(\mathbf{x}) \quad \forall j \in A. \quad (6.5)$$

For matrix $M^{\bar{j},q}$ is nonnegative and $M_{i_q, \bar{j}}^{\bar{j},q}$ is finite for all $j \in A$, (6.5) implies

$$-\infty < \sum_{j \in A} M_{i_q, j}^{\bar{j},q} (v_j(\bar{\mathbf{x}}) - v_j(\mathbf{x})) \leq 0.$$

The latter combined with assumption (2) thus gives

$$\lim_{q \rightarrow \infty} \frac{\sum_{j \in A} M_{i_q, j}^{\bar{j},q} (v_j(\bar{\mathbf{x}}) - v_j(\mathbf{x}))}{M_{i_q, \bar{j}}^{\bar{j},q}} = 0.$$

From (6.4) we therefore deduce $v_{\bar{j}}(\mathbf{x}) - v_{\bar{j}}(\bar{\mathbf{x}}) \geq 0$ and

$$v_{\bar{j}}(\bar{\mathbf{x}}) \leq v_{\bar{j}}(\mathbf{x}) \quad \forall j \in B. \quad (6.6)$$

Both (6.5) and (6.6) yield $\mathbf{v}(\bar{\mathbf{x}}) \leq \mathbf{v}(\mathbf{x})$. For $\mathbf{f}_{\mathcal{U}^S}^{\mathbf{v}\text{-AiO}}(\bar{\mathbf{x}}) \neq \mathbf{f}_{\mathcal{U}^S}^{\mathbf{v}\text{-AiO}}(\mathbf{x})$ because of (6.3), we obtain $\mathbf{v}(\bar{\mathbf{x}}) \leq \mathbf{v}(\mathbf{x})$, a contradiction to $\mathbf{x} \in E(X, \mathbf{v})$. The proofs for strictly and weakly efficient solutions are analogous. $\square$

Corollary 6.5. *Let $\mathcal{S} \subseteq \mathcal{M}^{\geq}$. If $\mathcal{S}$ satisfies the assumptions of Proposition 6.4 then $\mathcal{H}^{\mathcal{S}}$ is vector-AiO-basis-preserving.*

Proof. The assumptions of Proposition 6.4 guarantee that the assumption of Proposition 6.2 is satisfied. Propositions 6.2, 6.3.(i), and 6.4 make $\mathcal{H}^{\mathcal{S}}$ be vector-AiO-basis-preserving. $\square$

For the AaO approach with nonnegative-coefficient linear operators, we present the same result as Proposition 4.3 but under less restrictive assumptions.

Proposition 6.6. *Let $\mathcal{S} \subseteq \mathcal{M}^{\geq}$ and $\mathcal{U}^{\mathcal{S}}$ be the associated uncertainty set (6.2). If for every $j \in [b]$ there exists a matrix in $\mathcal{S}$ with nonzero j^{th} column then $E^{\text{v-AaO}}(X, \mathbf{f}, \mathcal{U}^{\mathcal{S}}) \subseteq E(X, \mathbf{v})$.*

Proof. Let $\mathbf{x} \in E^{\text{v-AaO}}(X, \mathbf{f}, \mathcal{U}^{\mathcal{S}})$, that is, $\mathbf{x} \in E(X, M\mathbf{v})$ for every $M \in \mathcal{S}$. Suppose $\mathbf{x} \notin E(X, \mathbf{v})$. There then exists $\bar{\mathbf{x}} \in X$ such that $\mathbf{v}(\bar{\mathbf{x}}) \leq \mathbf{v}(\mathbf{x})$. Let $j \in [b]$ be such that $v_j(\bar{\mathbf{x}}) < v_j(\mathbf{x})$. For every matrix $\bar{M} \in \mathcal{S}$ such that its j^{th} column has a nonzero entry, we thus have $\bar{M}\mathbf{v}(\bar{\mathbf{x}}) \leq \bar{M}\mathbf{v}(\mathbf{x})$, a desired contradiction. $\square$

For strictly/weakly efficiency, Propositions 4.4 and the order-preserving property of the operators in $\mathcal{H}^{\geq}$ yield the following.

Corollary 6.7. *Let $\mathcal{S} \subseteq \mathcal{M}^{\geq}$ and $\mathcal{U}^{\mathcal{S}}$ be the associated uncertainty set (6.2). We have*

- (i) $E_s^{\text{v-AaO}}(X, \mathbf{f}, \mathcal{U}^{\mathcal{S}}) \subseteq E_s(X, \mathbf{v})$,
- (ii) $E_w^{\text{v-AaO}}(X, \mathbf{f}, \mathcal{U}^{\mathcal{S}}) \subseteq E_w(X, \mathbf{v})$ if at least one matrix in $\mathcal{S}$ has no all-zero rows.

Based on the presented results, the classes of operators given below are vector-AiO-basis preserving.

Stochastic matrices. To represent the concept of p convex combinations on the basis function $v_1, \dots, v_b$, we use the σ -row-stochastic matrices and σ -column-stochastic matrices.

Definition 6.8. Let $\sigma \in \mathbb{R}_{\geq}^p (\mathbb{R}_{\geq}^b)$. A matrix $M \in \mathcal{M}^{\geq}$ is called σ -row(column)-stochastic if for every row i (column j) of M , all the entries sum to σ_i (σ_j), that is, $\sum_{j=1}^b m_{ij} = \sigma_i$ for every $i \in [p]$ ($\sum_{i=1}^p m_{ij} = \sigma_j$ for every $j \in [b]$).

Notice that for a σ -row-stochastic matrix M , if the j^{th} component of σ equals 1 then the j^{th} component of vector $M\mathbf{v}$ is a convex combination of the components of $\mathbf{v}$.

Definition 6.9. Let $\sigma_r \in \mathbb{R}_{\geq}^p$ and $\sigma_c \in \mathbb{R}_{\geq}^b$. A matrix $M \in \mathcal{M}^{\geq}$ is called (σ_r, σ_c) -stochastic if it is both σ_r -row-stochastic and σ_c -column-stochastic.

Let $\mathcal{M}^{\sigma\text{-rs}}$, $\mathcal{M}^{\sigma\text{-cs}}$, and $\mathcal{M}^{(\sigma_r, \sigma_c)\text{-s}}$ denote the set of σ -row-stochastic, σ -column-stochastic, and (σ_r, σ_c) -stochastic matrices, respectively. Note that $\mathcal{M}^{(\sigma_r, \sigma_c)\text{-s}} = \mathcal{M}^{\sigma_r\text{-rs}} \cap \mathcal{M}^{\sigma_c\text{-cs}}$. One can see that $\mathcal{M}^{(\sigma_r, \sigma_c)\text{-s}}$ fulfills the assumption in Proposition 6.1. We obtain the following corollary to Proposition 6.1.

Corollary 6.10. *Let $\sigma_r \in \mathbb{R}_{\geq}^p$ and $\sigma_c \in \mathbb{R}_{\geq}^b$. The sets $\mathcal{H}^{\mathcal{M}^{\sigma_r\text{-rs}}}$, $\mathcal{H}^{\mathcal{M}^{\sigma_c\text{-cs}}}$, and $\mathcal{H}^{\mathcal{M}^{(\sigma_r, \sigma_c)\text{-s}}}$ are vector-AiO-basis-preserving.*

Convex and/or conical combination. By Proposition 6.1 and Corollary 6.10, sets of linear operators defined by matrices inducing either convex combinations of $\mathbf{v}$ (i.e., $\mathcal{M}^{\sigma\text{-cs}}$ with $\sigma = \mathbf{1}$), conical combinations of $\mathbf{v}$ (i.e., $\mathcal{M}^{\geq}$), or bounded conical combinations (i.e., $\mathcal{M}^{\text{bc}} := \{M \in \mathcal{M}^{\geq}, \sum_{j=1}^b m_{ij} \leq 1, \forall i \in [p]\}$) are vector-AiO-basis-preserving.

For sets of linear operators combining convex and conical combinations of basis functions, their vector-AiO-basis-preserving-ness relates to Corollary 6.5 instead. Let $q \in [b]$ and $\mathcal{M}^{\text{c+c}} \subseteq \mathcal{M}^{\geq}$ be the

set of $p \times b$ matrices M satisfying $\sum_{j=1}^q m_{ij} = 1$ for every $i \in [p]$. The linear operators represented by the matrices in $\mathcal{M}^{c+c}$ correspond to sums of convex combinations of the first q basis functions and conical combinations of the last $b - q$ basis functions. Since the matrices in $\mathcal{M}^{c+c}$ have no all-zero rows, these linear operators are both order-preserving and strictly order-preserving. Additionally, we observe the following.

Corollary 6.11. *The set of linear operators represented by $\mathcal{M}^{c+c}$ is vector-AiO-basis preserving.*

Proof. Let $[b]$ be partitioned into $\{A, B\}$ where $A = \{1, \dots, q\}$ and $B = \{q + 1, \dots, b\}$. For every index in B , there exists a sequence of matrices in $\mathcal{M}^{c+c}$ such that all assumptions of Proposition 6.4 hold. Therefore, the result is a consequence of Corollary 6.5. $\square$

Making use of these linear operators, we construct the uncertainty set

$$\mathcal{U}^{c+c} := \{\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^p, \mathbf{f} = M^{c+c} \mathbf{v}, M^{c+c} \in \mathcal{M}^{c+c}\}, \quad (6.7)$$

for which the following holds directly from Proposition 6.6.

Corollary 6.12. *Let $\mathcal{U}$ be the uncertainty set (6.7). Then $E_{(\cdot/s/w)}^{v-AaO}(X, \mathbf{f}, \mathcal{U}^{c+c}) \subseteq E_{(\cdot/s/w)}(X, \mathbf{v})$.*

Entrywise addition and/or multiplication. Let $\mathcal{S}^{\theta, \tau} \subseteq \mathbb{R}^b \times \mathbb{R}^b$ and define the set of operators

$$\mathcal{H}^{\theta, \tau} := \{\mathbf{h} : \mathbb{R}^b \rightarrow \mathbb{R}^b : \mathbf{h}(\mathbf{z}) = \boldsymbol{\theta} \odot \mathbf{z} + \boldsymbol{\tau}, (\boldsymbol{\theta}, \boldsymbol{\tau}) \in \mathcal{S}^{\theta, \tau}\},$$

where $+$ and $\odot$ denote respectively the vector addition and the Hadamard multiplication of two vectors. Notice that if $\mathcal{S}^{\theta, \tau} = \mathcal{S}^{1, \tau} := \{(\mathbf{1}, \boldsymbol{\tau}) : \boldsymbol{\tau} \in \mathbb{R}^b\}$ and $\mathcal{S}^{\theta, \tau} = \mathcal{S}^{\theta, 0} := \{(\boldsymbol{\theta}, \mathbf{0}) : \boldsymbol{\theta} \in \mathbb{R}^b\}$ where $\mathbf{1}$ and $\mathbf{0}$ are the b -dimensional all-one and all-zero vectors, the operators in $\mathcal{H}^{\theta, \tau}$ represent the entrywise vector addition and Hadamard multiplication, respectively.

Given $(\boldsymbol{\theta}, \boldsymbol{\tau}) \in \mathcal{S}^{\theta, \tau}$, the associated operator $\mathbf{h}$ of $\mathcal{H}^{\theta, \tau}$ has the following matrix representation $\mathbf{h}(\mathbf{z}) = \begin{bmatrix} T^{\boldsymbol{\theta}} & \boldsymbol{\tau} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ 1 \end{bmatrix}$, where

$$T^{\boldsymbol{\theta}} := \begin{bmatrix} \theta_1 & & & \\ & \theta_2 & & 0 \\ & & \ddots & \\ 0 & & & \theta_b \end{bmatrix}.$$

is a $b \times b$ diagonal matrix whose i th main diagonal element is the i th component of $\boldsymbol{\theta}$, $i \in [b]$. If $\mathcal{S}^{\theta, \tau} \subseteq (\mathbb{R}_{>}^b \times \mathbb{R}_{\geq}^b) \times \mathbb{R}^b$, then the functions in $\mathcal{H}^{\theta, \tau}$ are (strictly) order-preserving. From Proposition 6.1, we observe the following.

Corollary 6.13. *Let $\mathcal{S}^{\theta, \tau} \subset \mathbb{R}_{\geq}^b \times \mathbb{R}^b$. If there does not exist $i \in [b]$ such that $\theta_i = 0$ for every $(\boldsymbol{\theta}, \boldsymbol{\tau}) \in \mathcal{S}^{\theta, \tau}$, the set $\mathcal{H}^{\theta, \tau}$ is vector-AiO-basis-preserving.*

Multiple entrywise addition and/or multiplication. Let $\mathcal{S}^{m\theta, \tau} \subseteq \mathbb{R}^{b'} \times \mathbb{R}^{b''}$ for given $b', b'' \in \mathbb{N}$, and define the set of operators $\mathcal{H}^{m\theta, \tau} := \{\mathbf{h} : \mathbb{R}^b \rightarrow \mathbb{R}^{bb'b''} : \mathbf{h}(\mathbf{z}) = \boldsymbol{\theta} \otimes \mathbf{z} \oplus \boldsymbol{\tau}, (\boldsymbol{\theta}, \boldsymbol{\tau}) \in \mathcal{S}^{m\theta, \tau}\}$, where $\oplus$ and $\otimes$ denote respectively the multiple entrywise addition of two vectors (i.e., addition of every entry of the first vector to each entry of the second vector) and the Kronecker multiplication.

Given $(\boldsymbol{\theta}, \boldsymbol{\tau}) \in \mathcal{S}^{m\theta, \tau}$, let $T^{m\theta} := [\mathbf{1}_{b''} \otimes \boldsymbol{\theta} \otimes I_b]$ be the $(bb'b'') \times b$ matrix obtained by duplicating b'' times the $(b'b) \times b$ matrix,

$$\begin{bmatrix} \theta_1 I_b \\ \theta_2 I_b \\ \vdots \\ \theta_{b'} I_b \end{bmatrix},$$

and let $\mathbf{t}^\tau := \tau \otimes \mathbf{1}_{bb'}$ be the $(bb'b'')$ -dimensional vector obtained by duplicating (bb') times vector τ , where $\mathbf{1}_{b''}$ and $\mathbf{1}_{bb'}$ are respectively the b'' - and (bb') -dimensional all-one vectors and I_b is the identity matrix of size b . The operator $\mathbf{h}$ of $\mathcal{H}^{m\theta, \tau}$ associated with (θ, τ) has the following matrix representation

$$\mathbf{h}(\mathbf{z}) = \begin{bmatrix} T^{m\theta} & \mathbf{t}^\tau \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ 1 \end{bmatrix}.$$

If $\mathcal{S}^{m\theta, \tau} \subseteq (\mathbb{R}_{>}^{b'} \times \mathbb{R}^{b''}) \mathbb{R}_{\leq}^{b'} \times \mathbb{R}^{b''}$, then the functions in $\mathcal{H}^{m\theta, \tau}$ are (strictly) order-preserving. From Proposition 6.1, we observe the following.

Corollary 6.14. *Let $\mathcal{S}^{m\theta, \tau} \subset \mathbb{R}_{\geq}^b \times \mathbb{R}^b$. The set $\mathcal{H}^{m\theta, \tau}$ is vector-AiO-basis-preserving.*

Matrix-valued functions. All the linear operators considered in this section are associated with non-negative real matrices. By considering matrices whose entries are real-valued uni-variate functions instead, similar results can be derived provided these functions are (strictly) increasing. Matrix-valued functions are used in theory of linear systems as transfer matrices to model relationships between multiple inputs and outputs of the system. For example, impedance or admittance matrices with components being functions of frequency play this role in mathematical modeling of electrical circuits [10].

As a simple example in the context of this study, let $\mathcal{S} \subseteq \mathbb{R}$ and $\mathcal{D}^{\mathcal{S}}$ denote the set of diagonal matrices $M^\alpha(\mathbf{z}) = [m_{ii}^\alpha(z)]_{i \in [b]}$ whose diagonal entries are the real-valued function $m_{ii} : z \mapsto z^\alpha, i \in [b]$, where $z \in \mathbb{R}$ and $\alpha \in \mathcal{S}$.

Let $\mathcal{H}^\alpha := \{\mathbf{h} : \mathbb{R}^b \rightarrow \mathbb{R}^b, \mathbf{h}(\mathbf{z}) = \mathbf{z}^{\odot \alpha}, \alpha \in \mathcal{S}\} = \{\mathbf{h} : \mathbb{R}^b \rightarrow \mathbb{R}^b, \mathbf{h}(\mathbf{z}) = M^\alpha(\mathbf{z}), M^\alpha \in \mathcal{D}^{\mathcal{S}}\}$ be the set of operators of entrywise power of a vector, where $\mathbf{z}^{\odot \alpha}$ denotes the α -th Hadamard power of $\mathbf{z}$. If $\mathcal{S} \subset (\mathbb{R}_{>})\mathbb{R}_{\geq}$ then the functions in $\mathcal{H}^\alpha$ are (strictly) order-preserving on $(\mathbb{R}_{>}^b)\mathbb{R}_{\geq}^b$. If $\mathcal{S} \subset (\mathbb{R}_{>}^{\text{odd}})\mathbb{R}_{\geq}^{\text{odd}}$ or $\mathcal{S} \subset (\mathbb{R}_{<}^{\text{odd}})\mathbb{R}_{\leq}^{\text{odd}}$ then the functions in $\mathcal{H}^\alpha$ are (strictly) order-preserving on $(\mathbb{R}_{<}^b)\mathbb{R}_{\leq}^b$, where $(\mathbb{R}_{>}^{\text{odd}})\mathbb{R}_{\geq}^{\text{odd}}$ denotes the set of odd (positive) nonnegative real numbers. As long as set $\mathcal{S}$ satisfies the aforementioned assumptions and function $z \mapsto z^\alpha$ is increasing for $z \in \mathbb{R}$ and $\alpha \in \mathcal{S}$, Proposition 4 implies that $\mathcal{H}^\alpha$ is vector-AiO-basis-preserving.

Linear operators can also be associated with complex systems in the form of the union of linear systems. If all the systems in the union are governed by sets of order-preserving operators while one of these sets is vector vector-AiO-basis-preserving, then the union of the operator sets is vector-AiO-basis-preserving.

6.1. Scalar-valued uncertainty. We continue the consideration of linear operators for the scalar-uncertainty case. As previously established in (2.5), scalar-valued uncertainty can be viewed as a special case of vector-valued uncertainty. For every $i \in [p]$, we consider the sets $\mathcal{S}_i \subseteq \mathbb{R}_{\geq}^{b_i}$ and $\mathcal{H}_i^{\mathcal{S}_i} := \{h_i : \mathbb{R}^{b_i} \rightarrow \mathbb{R}, h_i(\mathbf{z}_i) = \mathbf{m}_i^T \mathbf{z}_i, \mathbf{m}_i \in \mathcal{S}_i\}$. These p sets of linear operators make up the set $\tilde{\mathcal{H}}^{\mathcal{S}} := \{\tilde{\mathbf{h}} : \mathbb{R}^b \rightarrow \mathbb{R}^p : \tilde{\mathbf{h}}(\mathbf{z}) = \tilde{M}\mathbf{z}, \tilde{M} \in \tilde{\mathcal{M}}^{\mathcal{S}}\}$, where $\mathcal{S} := \mathcal{S}_1 \times \cdots \times \mathcal{S}_p$, $\mathbf{z} = [\mathbf{z}_1 \ \cdots \ \mathbf{z}_p]^T$, and $\tilde{\mathcal{M}}^{\mathcal{S}}$ is the set of nonnegative (and nonzero) matrices

$$\tilde{M} = \begin{bmatrix} \mathbf{m}_1^T & & 0 \\ & \mathbf{m}_2^T & \\ 0 & & \ddots \\ & & & \mathbf{m}_p^T \end{bmatrix} \quad (6.8)$$

with $\mathbf{m}_i \in \mathcal{S}_i$, for every $i \in [p]$. Let

$$\mathcal{U}_i^{\mathcal{S}_i} := \{f_i : \mathbb{R}^n \rightarrow \mathbb{R}, f_i = \mathbf{m}_i^T \mathbf{v}_i, \mathbf{m}_i \in \mathcal{S}_i\} = \{f_i^{k_i} : \mathbb{R}^n \rightarrow \mathbb{R}, f_i^{k_i} = \mathbf{m}_i^{k_i^T} \mathbf{v}_i, k_i \in \mathcal{K}_i\} \quad (6.9)$$

for every $i \in [p]$, and

$$\mathcal{U}^{\mathcal{S}} := \{\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^p : \mathbf{f} = \tilde{M}\mathbf{z}, \tilde{M} \in \tilde{\mathcal{M}}\} = \{\mathbf{f}^{\tilde{k}} : \mathbb{R}^n \rightarrow \mathbb{R}^p, \mathbf{f}^{\tilde{k}} = \tilde{M}^{\tilde{k}}\mathbf{z}, \tilde{k} \in \tilde{\mathcal{K}}\}, \quad (6.10)$$

where $\tilde{k} := (k_1, \dots, k_p) \in \tilde{\mathcal{K}} := \mathcal{K}_1 \times \dots \times \mathcal{K}_p$. There exists an equivalence between set $\tilde{\mathcal{H}}^{\mathcal{S}}$ being vector-AiO-basis-preserving and sets $\mathcal{H}_1^{\mathcal{S}_1}, \dots, \mathcal{H}_p^{\mathcal{S}_p}$ being scalar-AiO-basis-reserving as stated below.

Proposition 6.15. *Let $\mathcal{S} = \mathcal{S}_1 \times \dots \times \mathcal{S}_p$ with $\mathcal{S}_i \subseteq \mathbb{R}^{b_i}$ for every $i \in [p]$. $\tilde{\mathcal{H}}^{\mathcal{S}}$ is vector-AiO-basis-preserving if and only if $\mathcal{H}_1^{\mathcal{S}_1}, \dots, \mathcal{H}_p^{\mathcal{S}_p}$ are scalar-AiO-basis-preserving.*

Proof. From (6.8), all the entries of vector $\left[\mathbf{f}_{\mathcal{U}_1^{\mathcal{S}_1}}^{\mathcal{S}\text{-AiO}} \dots \mathbf{f}_{\mathcal{U}_p^{\mathcal{S}_p}}^{\mathcal{S}\text{-AiO}} \right]^T$ correspond to the entries of vector $\mathbf{f}_{\mathcal{U}^{\mathcal{S}}}^{\mathcal{V}\text{-AiO}}$, and vice versa. Hence, $E_{(\cdot/s/w)}(X, \mathbf{f}_{\mathcal{U}^{\mathcal{S}}}^{\mathcal{V}\text{-AiO}}) = E_{(\cdot/s/w)} \left(X, \left[\mathbf{f}_{\mathcal{U}_1^{\mathcal{S}_1}}^{\mathcal{S}\text{-AiO}} \dots \mathbf{f}_{\mathcal{U}_p^{\mathcal{S}_p}}^{\mathcal{S}\text{-AiO}} \right]^T \right)$. $\square$

Propositions 2.3, 6.1, and 6.15 allow us to state sufficient conditions for the scalar-AiO-basis-preserving property of p scalar-valued operator sets $\mathcal{H}_1^{\mathcal{S}_1}, \dots, \mathcal{H}_p^{\mathcal{S}_p}$.

Proposition 6.16. *For every $i \in [p]$, let $\mathcal{S}_i \subseteq \mathbb{R}_{\geq}^{b_i}$. If for every $i \in [p]$ and every $j \in [b_i]$ there exists a vector $\mathbf{m}_i^j \in \mathcal{S}_i$ being a positive multiple of the b_i -dimensional vector $\mathbf{e}_j$, then the sets $\mathcal{H}_1^{\mathcal{S}_1}, \dots, \mathcal{H}_p^{\mathcal{S}_p}$ are scalar-AiO-basis-preserving.*

Using Propositions 6.2, 6.3, and 6.4 together with the connection between scalar-valued and vector-valued uncertainties, made up by the block-diagonal structure of matrices (6.8), we derive necessary conditions for scalar-valued operator sets to be scalar-AiO-basis-preserving. For every $i \in [p]$, let A_i be composed of all the integers j in $[b_i]$ for which there exists a vector in $\mathcal{S}_i$ that is a positive multiple of the b_i -dimensional vector $\mathbf{e}_j$, and let $B_i = [b_i] \setminus A_i$. We now can give a scalar-valued uncertainty equivalent of Corollary 6.5.

Corollary 6.17. *For every $i \in [p]$ let $\mathcal{S}_i \subseteq \mathbb{R}_{\geq}^{b_i}$ be an infinite set and (A_i, B_i) be a partition of $[b_i]$ as described above such that $A_i \neq \emptyset$. If for every $i \in [p]$, any one of the following two assumptions holds*

- (i) $B_i = \emptyset$,
- (ii) for every $\bar{j} \in B_i$ there exists a sequence $(\mathbf{m}^{\bar{j},1}, \mathbf{m}^{\bar{j},2}, \dots)$ of vectors in $\mathcal{S}_i$ such that
 - (1) the entries of every vector $\mathbf{m}^{\bar{j},q}$ in this sequence satisfy
 - (a) $m_j^{\bar{j},q} < \infty \quad \forall j \in A_i$
 - (b) $m_j^{\bar{j},q} = 0 \quad \forall j \in B_i \setminus \{\bar{j}\}$
 - (c) $m_{\bar{j}}^{\bar{j},q} > 0$
 - (2) $\lim_{q \rightarrow \infty} m_{\bar{j}}^{\bar{j},q} = \infty$

then the sets $\mathcal{H}_1^{\mathcal{S}_1}, \dots, \mathcal{H}_p^{\mathcal{S}_p}$ are scalar-AiO-basis-preserving.

In the scalar-AaO case, we first build upon the results obtained in the vector-AaO case.

Corollary 6.18. *Let the uncertainty sets be defined by (6.9) and (6.10). If for every $i \in [p]$*

- (i) $\mathcal{S}_i \subseteq \mathbb{R}_{\geq}^{b_i}$ is such that there exists a vector having a nonzero j^{th} component for every $j \in [b_i]$ then $E^{\mathcal{S}\text{-AaO}}(X, \mathbf{f}, \mathcal{U}^{\mathcal{S}}) \subseteq E(X, \mathbf{v})$,
- (ii) $\mathcal{S}_i \subseteq \mathbb{R}_{\geq}^{b_i}$, then $E_{s/w}^{\mathcal{S}\text{-AaO}}(X, \mathbf{f}, \mathcal{U}^{\mathcal{S}}) \subseteq E_{s/w}(X, \mathbf{v})$.

Linear operators with nonnegative coefficients allow us to obtain the reverse inclusion to that in Proposition 4.14 if the tuples in (4.3) are defined in a particular way. Let $g'_{ij} : \mathbb{R}^{b_i} \rightarrow \mathbb{R}$ be constant functions defined as $g'_{ij} \circ \mathbf{v}_i = \beta_{ij}$, where $\beta_{ij} \in \mathbb{R}$ for every $i \in [p]$ and every $j \in [b_i]$. Similar to (4.3) and the associated set of all tuples $\mathcal{W}$, using these functions we define the p sets

$$\mathcal{G}'_i := \{v_{i1} + \beta_{i1}, v_{i2} + \beta_{i2}, \dots, v_{ib_i} + \beta_{ib_i}\} \quad (6.11)$$

and the resulting set of all tuples $\mathcal{W}'$.

Proposition 6.19. *Let the uncertainty sets be defined by (6.9) and (6.10). Then $E_{(\cdot/s/w)}^{s\text{-AaO}}(X, \mathcal{W}') \subseteq E_{(\cdot/s/w)}^{s\text{-AaO}}(X, \mathbf{f}, \mathcal{U}^S)$.*

Proof. We prove the result for efficient solutions. Assume $\mathbf{x} \in E^{s\text{-AaO}}(X, \mathcal{W}')$ but $\mathbf{x} \notin E^{s\text{-AaO}}(X, \mathcal{U}^S)$. There then exist $\bar{k} \in \tilde{\mathcal{K}}$ and $\bar{\mathbf{x}} \in X$ such that $\mathbf{f}^{\bar{k}}(\bar{\mathbf{x}}) \leq \mathbf{f}^{\bar{k}}(\mathbf{x})$, or equivalently,

$$f_i^{\bar{k}_i}(\bar{\mathbf{x}}) \leq f_i^{\bar{k}_i}(\mathbf{x}) \quad \forall i \in [p] \quad (6.12)$$

with at least one strict inequality. Because of (6.9), inequalities (6.12) can be rewritten as

$$\sum_{j=1}^{b_i} m_{ij}^{\bar{k}_i} v_{ij}(\bar{\mathbf{x}}) \leq \sum_{j=1}^{b_i} m_{ij}^{\bar{k}_i} v_{ij}(\mathbf{x}) \quad \forall i \in [p] \quad (6.13)$$

with at least one strict inequality, where $\mathbf{m}_i^{\bar{k}_i} \in \mathcal{S}_i$ is such that $f_i^{\bar{k}_i} = \mathbf{m}_i^{\bar{k}_i T} \mathbf{v}_i$ for every $i \in [p]$, which is equivalent to

$$\sum_{j=1}^{b_i} m_{ij}^{\bar{k}_i} (v_{ij}(\bar{\mathbf{x}}) + \beta_{ij}) \leq \sum_{j=1}^{b_i} m_{ij}^{\bar{k}_i} (v_{ij}(\mathbf{x}) + \beta_{ij}) \quad \forall i \in [p],$$

with at least one strict inequality. Since $\mathbf{m}_i^{\bar{k}_i} \geq \mathbf{0}$ for every $i \in [p]$, then for every $i \in [p]$ there exists $j_i \in [b_i]$ such that

$$v_{ij_i}(\bar{\mathbf{x}}) + \beta_{ij_i} \leq v_{ij_i}(\mathbf{x}) + \beta_{ij_i}. \quad (6.14)$$

Because at least one inequality in (6.13) is strict, so is at least one of the p inequalities (6.14). Let $\mathbf{w}'$ be the vector in $\mathcal{W}'$ whose components are $v_{1j_1} + \beta_{1j_1}, \dots, v_{pj_p} + \beta_{pj_p}$. We obtain $\mathbf{w}'(\bar{\mathbf{x}}) \leq \mathbf{w}'(\mathbf{x})$, a contradiction to $\mathbf{x} \in E^{s\text{-AaO}}(X, \mathcal{W}')$. The proofs for strictly and weakly efficient solutions are analogous. $\square$

Combining Propositions 4.14 and 6.19 yields the desired equality of efficient sets.

Corollary 6.20. *Let the uncertainty set be defined by (6.9) and (6.10), and the assumptions of Proposition 4.14 hold. Then $E_{(\cdot/s/w)}^{s\text{-AaO}}(X, \mathbf{f}, \mathcal{U}^S) = E_{(\cdot/s/w)}^{s\text{-AaO}}(X, \mathcal{W}')$.*

In the next section, we go back to the four-bar truss design problem that has motivated us to highlight the main concepts we have presented in this study.

7. APPLICATION IN ENGINEERING DESIGN

UBOP (1.1) being in the form of UMOP (2.1) carries parameter uncertainty, $F \in \mathcal{F}$, in the deflection criterion. However, the loaded truss satisfies the superposition principle that the overall response of the structure subjected to multiple loads can be obtained by summing the responses calculated from each individual load. In other words, the combined effect of a force system acting on the truss is the sum of the effects of the individual forces. This property makes it possible to follow the reformulation in Section 2 and model the deflection over infinitely many loading scenarios, $F \in \mathcal{F}$, as the sum of the deflections in a finite number of scenarios that have been selected by the designer as primary for the truss and from which all other scenarios can be constructed.

Let vol denote the volume of the truss with the geometry $\mathbf{x}$ and Δ denote the deflection when a vector of forces $\mathbf{F}_j$ is applied. For every primary scenario, $j \in [s]$, given by a known vector of forces $\mathbf{F}_j$, calculate the deflection Δ_j as a function of only the truss geometry $\mathbf{x}$

$$\Delta_j(\mathbf{x}) = \Delta(\mathbf{x}, \mathbf{F}_j), \quad \forall j \in [s]. \quad (7.1)$$

The finite MOP (2.2) representing the truss design problem assumes the form

$$\min_{\mathbf{x} \in X} [\text{vol}(\mathbf{x}) \quad \Delta_1(\mathbf{x}) \quad \dots \quad \Delta_s(\mathbf{x})]^T. \quad (7.2)$$

The deterministic functions Δ_j are used as the basis functions to construct the uncertainty set $\mathcal{U}$ that contains all deflections the truss may assume. Since the truss is a linear system, these deflections are the output of unknown order-preserving operators acting on the basis functions, and therefore the truss has the inherent property to preserve or approximate the efficient set.

We choose the three primary scenarios for the four-bar truss depicted in Figure 2. The three loading scenarios are chosen to capture distinct responses on the substructure. The top scenario introduces combined bending and axial compression simulating complex multi-axial stresses like aerodynamic lift countering fuel weight (in case of an aircraft wing rib) or dynamic cable tension (in case of a crane jib); the middle scenario applies a uniform downward loading, representative of standard distributed gravity loads or self-weight; and the bottom scenario exerts a high-magnitude concentrated force at the mid-span, mimicking localized payload or equipment mounting point. The latter scenario distinguishes itself from the former by restricting the primary load path to fewer members of the structure.

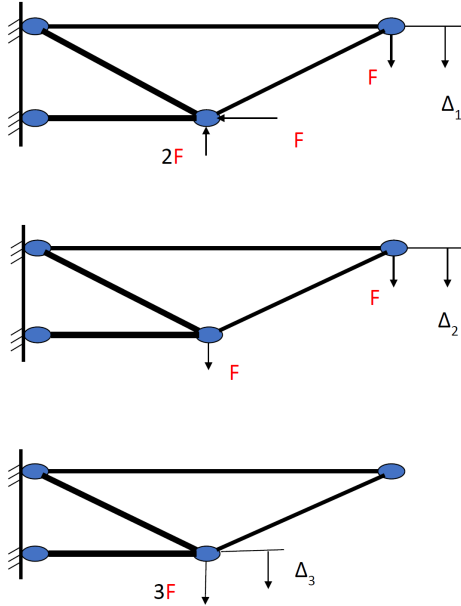


FIGURE 2. 4-bar truss with three primary scenarios

The feasible set is defined as

$$X = \{\mathbf{x} \in \mathbb{R}^4 : \epsilon \leq x_i \leq 1500\text{mm}^2, \sigma_i(\mathbf{x}) \leq 150\text{N/mm}^2, i \in [4]\}, \quad (7.3)$$

where x_i is the cross sectional area of bar i , σ_i is the stress assumed in bar i calculated as $\sigma_i(\mathbf{x}) = \frac{p_i}{x_i}$, where p_i is the (known) internal axial force in bar i , and $\epsilon = 10^{-6}\text{mm}^2$. Assuming the lengths of the bars are $l_1 = \sqrt{2}\text{m}$, $l_2 = 2\text{m}$, $l_3 = \sqrt{2}\text{m}$, $l_4 = 1\text{m}$, the volume of the truss is calculated as $\text{vol}(\mathbf{x}) = \sum_{i=1}^4 l_i x_i$. Taking the force $F = 1000\text{N}$, the deflection basis functions for the three scenarios are calculated

$$\Delta_1(\mathbf{x}) = \frac{1}{E} \left(\frac{2000\sqrt{2}}{x_1} + \frac{2000}{x_2} - \frac{2000\sqrt{2}}{x_3} + \frac{2000}{x_4} \right)$$

$$\Delta_2(\mathbf{x}) = \frac{1}{E} \left(\frac{2000\sqrt{2}}{x_1} + \frac{2000}{x_2} + \frac{4000\sqrt{2}}{x_3} + \frac{6000}{x_4} \right)$$

$$\Delta_3(\mathbf{x}) = \frac{1}{E} \left(\frac{6000\sqrt{2}}{x_3} + \frac{3000}{x_4} \right),$$

where E is the Young modulus of the selected material.

For illustration, we give two examples of uncertainty sets based on the operators presented in Section 6. For example, in the vector-valued model, the uncertainty set $\mathcal{U}$ can be defined according to (6.2), with

$$M = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & m_{22} & \cdots & m_{2(s+1)} \end{bmatrix} \in \mathcal{M}^{\geq}, \quad (7.4)$$

with $m_{2j} \leq \bar{m}_{2j}$, where $\bar{m}_{2j}$, for every $j \in \{2, \dots, s+1\}$, is a constant selected by the designer who exercises the knowledge and experience to determine the maximum allowable deflection for the truss. In the scalar-valued model, the uncertainty can be defined according to (2.5), where $\mathcal{U}_1 := \{\text{vol}(\mathbf{x})\}$ and $\mathcal{U}_2$ is defined as

$$\mathcal{U}_2 := \left\{ \Delta : \mathbb{R}^n \rightarrow \mathbb{R}, \Delta(\mathbf{x}) = \sum_{j=1}^s m_{2j} \Delta_j(\mathbf{x}), \mathbf{0} \leq \mathbf{m}_2 \leq \bar{\mathbf{m}}_2 \right\}, \quad (7.5)$$

where $\bar{\mathbf{m}}_2 \in \mathbb{R}_{\geq}^s$ is also a constant vector selected by the designer. Note that matrix M defined in (7.4) has the form of matrix (6.8) obtained from the scalar model. Given the three representative loading conditions, $s = 3$, and the associated deflections, $\Delta_j, j \in [3]$, $\mathcal{U}_2$ is a set of other deflections for which the truss is being designed. For example, choosing $\mathbf{m}_2 = [1.0, 0.0, 1.0]$, the designer formulates a combined design scenario representing the superposition of multiple loading conditions. While the specific selection of coefficients is arbitrary for illustration purposes, the resulting load profiles serve as effective analogies to complex couplings. Physically, this combined scenario represents the structure under the simultaneous realization of independent stress factors. In the crane jib analogy, it simulates the simultaneous occurrence of dynamic loads and static weight. Similarly, for an aircraft wing rib, it captures the coupled response to aerodynamic forces and the forces due to a concentrated mass, such as an engine or landing gear assembly.

The two models of uncertainty sets are equivalent in the sense that for both these models UBOP (1.1) assumes the form of UMOP (2.6)

$$\left\{ \min_{\mathbf{x} \in X} \mathbf{f}(\mathbf{x}) = [\text{vol}(\mathbf{x}) \quad \Delta(\mathbf{x})]^T \right\}_{\Delta \in \mathcal{U}_2}, \quad (7.6)$$

where $\mathbf{f} : \mathbb{R}^n \rightarrow \mathbb{R}^2$. The transition from UBOP (1.1) to (7.6) for the truss design illustrates the general relationship between UMOP (2.1) and surrogate UMOP (2.6). Such a transition will be conceptually the same for each particular decision-making application but will have a different implementation.

Taking steel with $E = 2 \times 10^5 \text{N/mm}^2$, the three basis functions are used to construct the set of uncertain deflection functions in (7.5) for $n = 4, s = 3$, and $\bar{\mathbf{m}}_2 = [1 \ 1 \ 1]$.

Because the vector and scalar models of the UBOP are the same, the AiO vector and scalar approaches act in the same way and so do the AaO vector and scalar approaches. We therefore define $E(X, \mathbf{f}_{\mathcal{U}_2}^{\text{AiO}}) := E(X, \mathbf{f}_{\mathcal{U}_2}^{\text{v-AiO}}) = E(X, \mathbf{f}_{\mathcal{U}_2}^{\text{s-AiO}})$ and $E^{\text{AaO}}(X, \mathbf{f}, \mathcal{U}_2) := E^{\text{v-AaO}}(X, \mathbf{f}, \mathcal{U}_2) = E^{\text{s-AaO}}(X, \mathbf{f}, \mathcal{U}_2)$.

The UBOP to solve for robust efficient solutions is given by (7.3) - (7.5) - (7.6). From among many diverse methodologies to compute efficient sets [73, 19], we choose a genetic algorithm because it conveniently computes representations of efficient sets but also allows for plotting their Pareto set representations being the images in the objective space. Using the genetic algorithm NSGA II [72], we compute the set of efficient solutions, $E(X, \mathbf{v})$, to (7.2) given as

$$\min_{\mathbf{x} \in X} \mathbf{v}(\mathbf{x}) := [\text{vol}(\mathbf{x}) \quad \Delta_1(\mathbf{x}) \quad \Delta_2(\mathbf{x}) \quad \Delta_3(\mathbf{x})]^T. \quad (7.7)$$

We construct two families of BOPs using the functions in (4.3) and (6.11). Defining $g_{2j} \circ \Delta_j = g'_{2j} \circ \Delta_j = 0$ for every $j \in [3]$, we obtain $\mathcal{W} = \mathcal{W}'$ with elements $\mathbf{w}_\ell = [V(\mathbf{x}) \quad \Delta_\ell(\mathbf{x})]^T$, $\ell \in [3]$. We compute the efficient sets, $E(X, \mathbf{w}_\ell)$, to three BOPs,

$$\min_{\mathbf{x} \in X} \begin{bmatrix} \text{vol}(\mathbf{x}) \\ \Delta_1(\mathbf{x}) \end{bmatrix} \quad \text{and} \quad \min_{\mathbf{x} \in X} \begin{bmatrix} \text{vol}(\mathbf{x}) \\ \Delta_2(\mathbf{x}) \end{bmatrix} \quad \text{and} \quad \min_{\mathbf{x} \in X} \begin{bmatrix} \text{vol}(\mathbf{x}) \\ \Delta_3(\mathbf{x}) \end{bmatrix}, \quad (7.8)$$

and then compute $E^{\text{s-AaO}}(X, \mathcal{W}) = \bigcap_{\ell \in [3]} E(X, \mathbf{w}_\ell)$.

The computed efficient set, $E(X, \mathbf{v})$, contains 1000 solutions. The associated Pareto set, being the four-dimensional image in the objective space, is projected on three two-dimensional objective spaces (vol, Δ_j) for every $j \in [3]$. The projections are depicted in green, blue, and red in Figure 3. Among these efficient solutions, 439 solutions make up the set $E^{\text{s-AaO}}(X, \mathcal{W})$ since they assume identical values within a tolerance of $0.05 \times 1500 \text{mm}^2 = 75 \text{mm}^2$ in each BOP, which is acceptable for the four-bar truss design with the available data. In Figure 3, the objective space images of $E^{\text{s-AaO}}(X, \mathcal{W})$ are plotted in orange. The location of these image points confirms the containment $E^{\text{s-AaO}}(X, \mathcal{W}) \subset E(X, \mathbf{v})$, as given in Proposition 4.13.

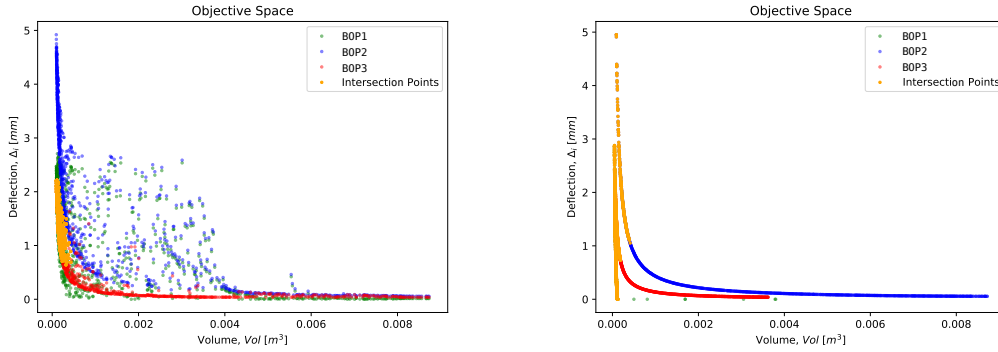


FIGURE 3. Green, blue, red: Three 2D projections of the Pareto set of (7.7) (left), and the Pareto sets of the BOPs in (7.8) (right); Orange: The objective space image of $E^{\text{s-AaO}}(X, \mathcal{W})$

Based on Proposition 6.1 and Corollary 3.5 for the vector case, or equivalently, Proposition 6.16 and Corollary 3.10 for the scalar case, $E(X, \mathbf{f}_{\mathcal{U}_2}^{\text{AiO}}) = E(X, \mathbf{v})$ and the computed designs are AiO-robust efficient to UBOP (7.3) - (7.5) - (7.6). Based on Proposition 6.6 for the vector case, or equivalently, Corollary 6.18(i) for the scalar case $E^{\text{AaO}}(X, \mathbf{f}, \mathcal{U}_2) \subseteq E(X, \mathbf{v})$ and the computed AiO-robust efficient set contains the AaO-robust efficient set. Finally, since the assumptions of Propositions 4.14 and 6.19 hold, based on Corollary 6.20, $E^{\text{AaO}}(X, \mathbf{f}, \mathcal{U}_2) = E^{\text{s-AaO}}(X, \mathcal{W})$ and the 439 computed designs are AaO-robust efficient to the UBOP.

This analysis may support the designer in choosing a final preferred robust-efficient design. Assume the designer has chosen some representative designs in $E(X, \mathbf{v})$ and $E^{\text{s-AaO}}(X, \mathcal{W})$ for consideration, as reported in Tables 1 and 2 respectively. Comparing the two groups of designs, note that the volume of designs in $E(X, \mathbf{v})$ is larger while the deflection is mostly smaller than of the designs in $E^{\text{s-AaO}}(X, \mathcal{W})$. However, since the AaO-robust efficient designs are also AiO-robust efficient, $E^{\text{AaO}}(X, \mathbf{f}, \mathcal{U}_2) \subseteq E(X, \mathbf{f}_{\mathcal{U}_2}^{\text{AiO}})$, the designer may be likely to choose a preferred design from the former.

TABLE 1. Representative designs in $E(X, \mathbf{v})$ and their volume and deflection values

$x_1(\text{mm}^2)$	$x_2(\text{mm}^2)$	$x_3(\text{mm}^2)$	$x_4(\text{mm}^2)$	vol(m^3)	$\Delta_1(\text{mm})$	$\Delta_2(\text{mm})$	$\Delta_3(\text{mm})$
18.7465	18.8700	28.3650	26.7203	1.31E-04	1.16	3.4042	2.0571
44.0208	20.5367	44.6587	30.0643	1.97E-04	0.8241	2.4394	1.4489
11.9832	9.8940	140.5877	147.8197	3.83E-04	0.1579	2.5950	0.4032
59.7584	17.6022	584.4378	344.9417	12.91E-04	0.8096	0.9401	0.1161
30.4920	15.9080	1416.652	1473.337	35.52E-04	1.0892	1.1327	0.4013

TABLE 2. Representative designs in $E^{\text{s-AaO}}(X, \mathcal{W})$ and their volume and deflection values

$x_1(\text{mm}^2)$	$x_2(\text{mm}^2)$	$x_3(\text{mm}^2)$	$x_4(\text{mm}^2)$	vol(m^3)	$\Delta_1(\text{mm})$	$\Delta_2(\text{mm})$	$\Delta_3(\text{mm})$
10.4253	7.4878	9.4390	10.8819	5.39E-05	2.1127	8.4454	5.8732
12.2396	8.9809	9.4372	11.7697	6.04E-05	1.6200	7.8150	5.7701
14.5639	10.0875	9.4730	13.4061	6.76E-05	1.2154	7.1859	5.5976
17.4390	12.0999	9.4901	16.4411	7.87E-05	0.7554	6.4425	5.3829
19.3068	15.5148	9.5030	21.2561	9.3E-05	0.3593	5.7648	5.1702

8. CONCLUSION

Being motivated by engineering systems characterized by multiple objectives and performing under uncertainty, and more specifically, by monotone or linear systems, we developed the model of UMOP under function uncertainty. In the proposed model, the uncertainty sets are made up by the set of all performance vectors of the system in all uncertainty scenarios.

Making use of mathematical optimization and engineering perspectives, and building on results in each of these fields, we proposed two robust counterparts, AiO and AaO, whose efficient solutions are robust efficient for the UMOP. While the counterparts account for all, possibly infinitely many scenarios, their efficient sets are preserved in or approximated by the efficient sets of associated finite MOPs. The properties of preservation and approximation require that the uncertain functions be constructed by order-preserving operators applied to the basis functions making up the objective functions of the associated finite MOPs. Because the performance of monotone or linear systems is compatible with the partial order, they inherently possess the properties to preserve or approximate AiO or AaO-robust efficient sets. For such systems, these sets can now be computed as the efficient sets of associated finite MOPs. Consequently, the proposed model of UMOP with function uncertainty can replace the UMOP with parameter uncertainty if the latter leads to computationally expensive or intractable counterparts but can be modeled as the former.

The AiO and AaO approaches come in two forms, vector and scalar. Being a special case of the vector formulation, the scalar one is more flexible because it allows for modeling the objective-wise uncertainty. The AiO approach preserves efficiency and may therefore seem more attractive. On the other hand, since the AaO-robust efficient solutions are also AiO-robust efficient, a final preferred solution selected by the DM may be AaO-robust efficient. In effect, the models complement each other and each of them can be used depending on circumstances.

It is of interest to extend this work by identifying nonlinear operators that are order preserving or by relaxing the preservation of the efficient set to allow a broader class of operators. This will bring uncertain multiobjective optimization even closer to applications in engineering and science.

STATEMENTS AND DECLARATIONS

The authors declare that they have no conflict of interest, and the manuscript has no associated data.

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