

OPTIMALITY CONDITIONS FOR WEAK EFFICIENCY IN VECTOR OPTIMISATION USING VECTORIAL PENALISATION

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ABSTRACT. We establish necessary optimality conditions for weakly efficient solutions of constrained, potentially nonconvex vector optimisation problems. The conditions are obtained by combining an unconstrained reformulation based on vectorial penalisation with nonlinear scalarisation and abstract subdifferential rules. Explicit multiplier conditions are derived for single and finite inequality systems, equality and mixed constraints, and closed convex feasible sets.

Keywords. Vector optimisation, weak efficiency, vectorial penalisation, optimality conditions, abstract subdifferential, Gerstewitz-functional

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1. INTRODUCTION

In our paper, we consider general constrained vector optimisation problems, where the objective function acts between real linear-topological spaces and the constraints are not supposed to be convex. The solution concept is given by a nontrivial pointed convex cone. Our aim is to derive the necessary and sufficient optimality conditions using the relationships between constrained and unconstrained vector optimisation problems and the concept of abstract subdifferentials. To do this, we will use new vectorial penalisation results derived in [8] and a nonlinear scalarisation method. We derive the necessary conditions for weakly efficient solutions for problems without convexity assumptions. In order to show the sufficient conditions, we suppose some generalised convexity assumptions concerning the objective function.

Recently, KKT-type necessary optimality conditions for nonsmooth vector equilibrium problems with set/cone/equality constraints are established in [18] using Clarke's and Michel–Penot's subdifferentials. Furthermore, weak subdifferentials and augmented normal cones have also been used to show the necessary (and sometimes sufficient) optimality conditions for efficient solutions in a nonconvex setting in [16]. Our main tool for deriving the optimality conditions for constrained vector optimisation problems is a vectorial penalisation approach combined with generalised subdifferentials. The necessary and sufficient optimality conditions for constrained vector optimisation using a vectorial penalisation approach and generalised subdifferentials are derived in [3].

The literature shows that KKT-like optimality conditions based on subdifferentials are proven tools for analysing efficient solutions in vector optimisation with nonconvex constraints, see [18, 12]. The use of generalised subdifferentials (Clarke/Michel–Penot/Mordukhovich/weak) allows these results to

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hold even when classical differentiability or convexity fails — a crucial advance given the prevalence of nonsmooth/nonconvex problems in applications. Penalisation techniques—including those based on variational analysis—offer alternative routes to optimality conditions that complement scalarisation approaches, see [3].

Despite this progress, deriving verifiable optimality conditions for weakly efficient solutions to nonconvex vector problems with constraints remains a complex task. In particular, one must combine scalarisation, penalisation, and generalised differentiation in such a way as to keep track of the specific assumptions used at every step.

Instead of working with one fixed subdifferential, we introduce an *abstract subdifferential* ∂ that satisfies a short list of axioms. This axiomatic approach offers two advantages: (i) The resulting optimality conditions apply to every specific subdifferential that satisfies the axioms, and (ii) it clearly identifies which properties of a subdifferential are indispensable and which are merely useful. In this way, it is possible to determine the “most cost-effective” subdifferential that is sufficient for a given class of problems.

The aim of the paper is to combine three principles: nonlinear scalarisation for nonconvex weak minimality, abstract vectorial penalisation for handling nonconvex feasible sets, and axiomatic subdifferential calculus for deriving optimality conditions without committing to one particular generalised derivative. After recalling the basic notions of weak efficiency, we formulate the penalisation assumptions and establish the corresponding relationships between constrained and unconstrained vector problems. We then introduce the abstract subdifferential framework and derive necessary optimality conditions as well as sufficient conditions and characterisations for weakly efficient solutions. In comparison with known results in the literature, especially in the paper [9, Theorem 4.1], we derive our new necessary optimality conditions for weakly efficient elements of constrained vector optimisation problems in a more general framework with an objective function acting between linear-topological spaces under more useful assumptions regarding the abstract subdifferential and regarding the lower-level sets / level lines of the penalisation function.

Our paper is organised as follows: In Section 2, we lay out the mathematical foundations for our investigations. We introduce the vector optimisation problem under consideration, corresponding solution concepts, the vectorial penalisation approach and the concept of abstract subdifferentials. Furthermore, we review the nonlinear Gerstewitz scalarisation. The latter is especially useful when the ordering cone has an empty interior, but we also exploit it in the solid-cone case because it yields a Lipschitz scalarising function without convexity assumptions on the objective. Section 3 contains the main optimality theorems. We show a necessary and sufficient condition for weak efficiency in the unconstrained case. When the abstract subdifferential satisfies (S1)–(S3), we obtain Lagrangian multiplier rules for a scalarised objective function. In Section 4, we give some conclusions.

2. BASICS

We use the notations $\mathbb{R}_{\geq} := [0, \infty)$, $\mathbb{R}_{\geq}^n := [0, \infty)^n$ and

$$\Delta_n := \left\{ x \in \mathbb{R}_{\geq}^n \left| \sum_{i=1}^n x_i = 1 \right. \right\}$$

for the simplex in $\mathbb{R}^n$. For a set I , its cardinality is denoted by $|I|$. For $\tau \in \{\text{loc}, \text{glob}\}$, we use the notation

$$\mathfrak{x} \in \tau\text{-argmin}_{x \in X} \phi(x)$$

to mean that $\mathfrak{x}$ is a local or global minimum of ϕ over X , respectively. We usually write argmin instead of $\operatorname{glob-argmin}$.

Throughout, $A \subset B$ denotes strict inclusion, whereas $A \subseteq B$ allows equality.

Let X be a real linear space. For sets $A, B \subseteq X$, we write $A + B$ for their Minkowski sum and define $A - B$ analogously. If $B = \{b\}$ is a singleton, we simply write $A + b := A + \{b\}$. For $\Lambda \subseteq \mathbb{R}$, we set

$$\Lambda A := \{ \lambda a \mid \lambda \in \Lambda, a \in A \},$$

and, for $\Lambda = \{\lambda\}$, write $\lambda A := \{\lambda\}A$. For $A \subseteq X$, we define the binary relations $\leq_A$ and $<_A$ on X by

$$x \leq_A y \iff y - x \in A, \quad x <_A y \iff y - x \in \operatorname{int}(A).$$

A subset $C \subseteq X$ is called a cone if $\mathbb{R}_{\geq} C = C$. A cone C is called pointed if $C \cap (-C) = \{0_X\}$ and proper if $C \neq X$.

If X is a real topological vector space, then X^* denotes its **topological dual space**. For $A \subseteq X$, we denote by $\operatorname{int}(A)$, $\operatorname{bd}(A)$ and $\operatorname{cl}(A)$ the interior, boundary and closure of A , respectively. For a set $A \subseteq X$, we define its **topological dual cone** by

$$A^{\geq} := \{ x^* \in X^* \mid x^*(a) \geq 0 \text{ for all } a \in A \}.$$

In particular, if X is locally convex and $C \subseteq X$ is a closed convex pointed cone with $\operatorname{int}(C) \neq \emptyset$, then $C^{\geq} \setminus \{0_{X^*}\} \neq \emptyset$. If Y is another real topological vector space, then the product space $X \times Y$ is always endowed with the product topology.

Throughout the paper we consider the following setting:

$$\begin{cases} X, Y \text{ real linear-topological spaces,} \\ f : X \rightarrow Y, \\ \{0_Y\} \neq C \subset Y \text{ pointed, convex cone,} \\ \emptyset \neq S \subseteq X \end{cases} \quad (A)$$

the constrained optimisation problem

$$\operatorname{argmin}_{x \in S} f(x). \quad \text{VOP}(f, S, C)$$

The set of **efficient solutions** of $\text{VOP}(f, S, C)$ is defined by

$$\operatorname{Eff}(f, S, C) := \{ \mathfrak{x} \in S \mid \nexists x \in S : f(x) \leq_C f(\mathfrak{x}), f(x) \neq f(\mathfrak{x}) \}.$$

If additionally $\operatorname{int}(C) \neq \emptyset$, then the set of **weakly efficient solutions** of $\text{VOP}(f, S, C)$ is defined by

$$\operatorname{WEff}(f, S, C) := \{ \mathfrak{x} \in S \mid \nexists x \in S : f(x) <_C f(\mathfrak{x}) \}.$$

We also need the following definitions and results.

Lemma 2.1 ([13, Prop. 7.1.2]). *Assume that $\mathbb{T}_1, \mathbb{T}_2$ are real linear-topological spaces and $A_1 \subseteq \mathbb{T}_1$, $A_2 \subseteq \mathbb{T}_2$ are convex sets. Then*

$$\operatorname{int}_{\mathbb{T}_1 \times \mathbb{T}_2}(A_1 \times A_2) = \operatorname{int}_{\mathbb{T}_1}(A_1) \times \operatorname{int}_{\mathbb{T}_2}(A_2).$$

Proposition 2.2. *Let $\mathbb{T}_1, \mathbb{T}_2$ be linear-topological spaces and $C_1 \subseteq \mathbb{T}_1$ and $C_2 \subseteq \mathbb{T}_2$ cones. Then $(\mathbb{T}_1 \times \mathbb{T}_2)^* = \mathbb{T}_1^* \times \mathbb{T}_2^*$ and $(C_1 \times C_2)^{\geq} = C_1^{\geq} \times C_2^{\geq}$.*

2.1. Abstract Penalisation functions. A powerful technique for converting constrained vector problems into unconstrained ones is the *vectorial penalisation approach* (see [8]). The essential idea is to append a scalar penalisation term ν to the vector objective f in (A), thereby forming an extended vector objective $f^\otimes := (f, \nu) : X \rightarrow Y \times \mathbb{R}$, where the image space is ordered by the product cone $C \times \mathbb{R}_\geq$, which is again a nontrivial, pointed, convex cone.

Under mild assumptions on ν , the weakly efficient set of $\text{VOP}(f, \mathcal{S}, C)$ can be expressed in terms of the weakly efficient sets of the two unconstrained problems $\text{VOP}(f, X, C)$ and $\text{VOP}(f^\otimes, X, C \times \mathbb{R}_\geq)$. This is especially valuable in the context of abstract subdifferentials because the unconstrained reformulation eliminates the normal-cone term from the optimality conditions.

From now on we impose the following assumptions:

$$\left\{ \begin{array}{l} \text{Assume (A),} \\ \nu : X \rightarrow \mathbb{R}, \\ f^\otimes := (f, \nu). \end{array} \right. \quad (\text{A}^\otimes)$$

Instead of fixing a concrete penalisation function, we consider general level-set properties. We first introduce the required notation.

Definition 2.3. Let $\mathcal{X}, \mathcal{Y}$ be two sets and let $S \subseteq \mathcal{X}$ and $\prec_A \subseteq \mathcal{Y} \times \mathcal{Y}$ be a binary relation. For a function $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $y \in \mathcal{Y}$, we define the $\prec_A$ -**level set** of f on S by

$$\text{lev}_{\prec_A}(S, f, y) := \{s \in S \mid f(s) \prec_A y\}.$$

Under assumption $(\text{A}^\otimes)$, the role of the penalty term ν is encoded by level-set conditions of the following form:

$$\forall x^0 \in \text{bd}_X(\mathcal{S}) : \text{lev}_\leq(X, \nu, \nu(x^0)) = \mathcal{S} \quad (\text{A1})$$

$$\forall x^0 \in \text{bd}_X(\mathcal{S}) : \text{lev}_=(X, \nu, \nu(x^0)) = \text{bd}_X(\mathcal{S}) \quad (\text{A2})$$

$$\forall x^0 \in \mathcal{S} : \text{lev}_=(X, \nu, \nu(x^0)) = \text{lev}_\leq(X, \nu, \nu(x^0)) = \mathcal{S} \quad (\text{A3})$$

$$\forall x^0 \in \mathcal{S} : \text{lev}_\leq(X, \nu, \nu(x^0)) \subseteq \mathcal{S} \quad (\text{A4})$$

$$\text{lev}_\leq(X, \nu, 0) = \mathcal{S} \quad (\text{A5})$$

$$\text{lev}_=(X, \nu, 0) = \text{bd}(\mathcal{S}) \quad (\text{A6})$$

$$\forall x^1 \in \text{bd}(\mathcal{S}) \exists x^2 \in \text{int}(\mathcal{S}) : [x^2, x^1] \subseteq \text{lev}_<(X, \nu, \nu(x^1)) \quad (\text{A7})$$

For simplicity, if ν satisfies one of the properties (A1), ..., (A7), we use the notation

$$\nu \in \mathcal{A}_i(\mathcal{S}, X) \quad (\text{for } i \in \{1, \dots, 7\}).$$

Some relationships between these properties and further results can be found in [8, Section 4.1]. It should be noted that if $\nu \in \mathcal{A}_1(\mathcal{S}, X)$, then $\mathcal{S}$ is closed and $\nu \in \mathcal{A}_4(\mathcal{S}, X)$. In general, if $\emptyset \neq \mathcal{S} \neq X$ then any $\nu \in \mathcal{A}_4(\mathcal{S}, X)$ is bounded from above on $\mathcal{S}$ and either $\text{lev}_\leq(X, \nu, \max_{s \in \mathcal{S}} \nu(s)) = \mathcal{S}$ or $\text{lev}_<(X, \nu, \sup_{s \in \mathcal{S}} \nu(s)) = \mathcal{S}$. Notice also that any $\nu \in \mathcal{A}_3(\mathcal{S}, X)$ satisfies $\nu \in \mathcal{A}_4(\mathcal{S}, X)$, and if $\nu \in \mathcal{A}_3(\mathcal{S}, X)$ is continuous, then $\mathcal{S}$ is closed. Properties (A5) and (A6) are closely related to (A1) and (A2) and usually occur as specific instances of these (see [10, Proposition 1.3.3]).

2.2. Abstract subdifferentials. We derive necessary optimality conditions for weakly efficient solutions of $\text{VOP}(f, \mathcal{S}, C)$ in terms of generalised subdifferentials. Rather than fixing a specific subdifferential, we use an operator ∂ that assigns to every function $\phi : X \rightarrow \mathbb{R}$ and every $x \in X$ a subset $\partial\phi(x) \subseteq X^*$.

For the operator ∂ we introduce the following assumptions:

(S1) If ϕ is convex, then $\partial\phi$ coincides with the Fenchel subdifferential, so

$$\partial\phi(x) = \partial^F\phi(x) := \{x^* \in X^* \mid \forall x' \in X : x^*(x' - x) + \phi(x) \leq \phi(x')\}.$$

(S2 $\vec{\tau}$) For $\tau \in \{\text{loc}, \text{glob}\}$, condition (S2 $\vec{\tau}$) denotes

$$\mathfrak{x} \in \tau - \operatorname{argmin}_{x \in X} \phi(x) \implies 0_{X^*} \in \partial\phi(\mathfrak{x}).$$

(S2 $\leftarrow\tau$) For $\tau \in \{\text{loc}, \text{glob}\}$, condition (S2 $\leftarrow\tau$) denotes

$$0_{X^*} \in \partial\phi(\mathfrak{x}) \implies \mathfrak{x} \in \tau - \operatorname{argmin}_{x \in X} \phi(x).$$

(S3) (**outer scalarisation chain rule**) If $\eta : Y \rightarrow \mathbb{R}$ is convex and $f : X \rightarrow Y$, then for every $x \in X$,

$$\partial(\eta \circ f)(x) \subseteq \bigcup_{y^* \in \partial\eta(f(x))} \partial(y^* \circ f)(x).$$

Remark 2.4. In other treatments of abstract subdifferential calculus (like [9, 4]), the outer scalarisation chain rule (S3) is often formulated only for mappings belonging to a distinguished function class $\mathcal{F}(X, Y)$. Such a class records the additional assumptions under which the chain rule is valid for the particular subdifferential and the spaces under consideration; these may include continuity, Lipschitz-type properties, compactness conditions or structural assumptions on the domain and image spaces. In this sense, $\mathcal{F}(X, Y)$ is a domain-stability assumption for the scalarisations used later. For instance, for the limiting (or Mordukhovich) subdifferential, if X is an Asplund space and Y is finite-dimensional, one may take $\mathcal{F}(X, Y)$ to be the class of locally Lipschitz mappings from X into Y . Indeed, every finite-valued convex function $\eta : Y \rightarrow \mathbb{R}$ is locally Lipschitz, and the limiting-subdifferential chain rule then gives precisely (S3) (see [5, Lemma 2.4]). This makes it possible to use the same abstract notation while retaining the sharper applicability conditions of individual subdifferentials.

The same refinement could be used here by restricting (S3) to an appropriate class of mappings and adding the corresponding membership assumptions whenever the rule is applied. For the present analysis, however, we suppress this additional notation and regard the applicability of the chain rule to the mappings under consideration as part of assumption (S3). Thus, the unrestricted formulation of the axiom is a notational simplification and is not intended to assert that every concrete subdifferential satisfies the chain rule for every mapping.

It is known that these axioms are fulfilled for several important subdifferentials under suitable assumptions on the spaces, functions and mappings involved.

The precise assumptions depend on the chosen subdifferential and its natural function class. Property (S1) holds for the standard subdifferentials considered here on continuous convex real-valued functions acting between Banach spaces, and the local Fermat rule (S2 $\vec{\text{loc}}$) is available for locally Lipschitz functions. For the limiting (or Mordukhovich) subdifferential, the exact outer scalarisation chain rule (S3) holds when X is an Asplund space, Y is finite-dimensional, $f : X \rightarrow Y$ is locally Lipschitz at the point under consideration and the convex outer function is locally Lipschitz at the corresponding image point (see [5, Lemma 2.4]). For the approximate (or Ioffe) subdifferential on Banach spaces X and Y , the same rule holds when f is strongly compactly Lipschitz at the point and the outer function is locally Lipschitz (see [5, Lemma 2.7]). The Clarke subdifferential generally satisfies (S3) only after taking the weak-star closed convex hull of the right-hand side (see [5, Section 3]).

Moreover, (S2 $\vec{\text{glob}}$) implies (S2 $\leftarrow\text{loc}$), while (S2 $\vec{\text{loc}}$) implies (S2 $\leftarrow\text{glob}$). For necessary optimality conditions we will not require (S2 $\leftarrow\tau$), which can give sufficient conditions.

We do not pursue the technical details of individual constructions here. Important examples include the Mordukhovich, Michel–Penot and Ioffe subdifferentials; see, for instance, [15] and the references cited above.

The following additional properties will be useful for further applications, although they are not needed for the main optimality theorem below. They still represent standard and natural requirements in subdifferential calculus (see Clarke [2, Section 2.3]). As before, each property may be restricted to suitable classes of functions adapted to the subdifferential under consideration. A common restriction, for example, is to require the functions to be Lipschitz continuous on a Banach space.

(S4) If ϕ is locally constant around x , then $\partial\phi(x) = \{0_{X^*}\}$.

(S5) **(sum rule)** For any two functions $\phi_1, \phi_2 : X \rightarrow \mathbb{R}$ and every $x \in X$, we have

$$\partial(\phi_1 + \phi_2)(x) \subseteq \partial\phi_1(x) + \partial\phi_2(x).$$

(S6) **(positive homogeneity)** For every function $\phi : X \rightarrow \mathbb{R}$, every $\alpha \in \mathbb{R}_{>}$ and every $x \in X$, we have

$$\partial(\alpha\phi)(x) \subseteq \alpha\partial\phi(x).$$

(S7) **(max rule)** For any functions $\phi_1, \dots, \phi_k : X \rightarrow \mathbb{R}$ and every $x \in X$, we have

$$\partial\left(\max_{i=1,\dots,k} \phi_i\right)(x) \subseteq \text{conv}\left(\bigcup_{i=1,\dots,k} \partial\phi_i(x)\right).$$

(S8) **(scalar composition chain rule)** For every $\phi : X \rightarrow \mathbb{R}$, every Lipschitz continuous $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ and every $x \in X$ with $\partial\phi(x) \neq \emptyset$, we have

$$\partial(\Psi \circ \phi)(x) \subseteq \partial\Psi(\phi(x)) \cdot \partial\phi(x).$$

Remark 2.5. Under (S1) the local-constant property (S4) and the positive homogeneity (S6) hold directly for convex functions. In fact, the latter holds with equality. The sum, max and chain rules are also standard rules of convex subdifferential calculus, but in general they require the usual convexity and qualification assumptions, such as suitable continuity or interiority conditions (see [19, Chapter 2]).

Notice that (S4) is satisfied if a local version of (S1) holds: If, for every function $\phi : X \rightarrow \mathbb{R}$ that is locally convex at $x \in X$, the subdifferential $\partial\phi(x)$ coincides with the Fenchel subdifferential and ϕ is locally constant at x , then $\partial\phi(x) = \{0_{X^*}\}$.

If (S2_{loc}[→]) holds and ϕ is locally constant around x , then $0_{X^*} \in \partial\phi(x)$. Hence (S2_{loc}[→]) gives the nonemptiness part of (S4), but not the reverse inclusion $\partial\phi(x) \subseteq \{0_{X^*}\}$.

The nonemptiness requirement in the scalar composition chain rule (S8) is necessary under the standard definition of scalar multiplication of sets used above. Indeed, for a constant outer function Ψ , the right-hand side contains $0\partial\phi(x)$, which is empty when $\partial\phi(x) = \emptyset$, whereas the left-hand side is the subdifferential of a constant function and therefore equals $\{0_{X^*}\}$ under (S1).

It is sufficient to require the max rule (S7) for two functions. The finite version then follows by induction.

As with the outer scalarisation chain rule (S3), the scalar composition chain rule (S8) may be restricted to a suitable class of outer functions adapted to the particular subdifferential under consideration. Requiring Ψ to be Lipschitz continuous is a natural choice because this is a standard setting for nonsmooth chain rules and includes the outer functions used below, such as the positive-part function and finite maxima. Other function classes may be used instead, provided that the corresponding scalar composition rule is available.

Proposition 2.6. *The assumptions introduced above have the following elementary consequences.*

- (1) Assume (S1) and the scalar composition chain rule (S8). If $\phi : X \rightarrow \mathbb{R}$ is locally Lipschitz continuous and $\partial\phi(x) \neq \emptyset$, then the max rule (S7) at x for the maximum of ϕ and one constant follows. (see ?? 3.3 (1))

- (2) Assume (S1), the outer scalarisation chain rule (S3) and the positive homogeneity (S6). Then, for every function $\phi : X \rightarrow \mathbb{R}$ and every $x \in X$ with $\partial\phi(x) \neq \emptyset$, the scalar composition chain rule (S8) holds for every convex, nondecreasing and Lipschitz continuous function $\Psi : \mathbb{R} \rightarrow \mathbb{R}$. If (S6) is strengthened by requiring the homogeneity inclusion for every $\alpha \in \mathbb{R} \setminus \{0\}$, the same conclusion holds for every convex Lipschitz continuous function $\Psi : \mathbb{R} \rightarrow \mathbb{R}$, without the monotonicity assumption.
- (3) Assume (S4) and the sum rule (S5). If $\phi, \psi : X \rightarrow \mathbb{R}$ coincide locally around x , then

$$\partial\phi(x) = \partial\psi(x).$$

Proof. (1). Let $\phi : X \rightarrow \mathbb{R}$ be locally Lipschitz continuous with $\partial\phi(x) \neq \emptyset$, and let $c \in \mathbb{R}$. Define $\Psi(t) := \max\{t, c\}$. The function Ψ is convex and Lipschitz continuous. Thus, by (S1), at $t < c$ one has $\partial\Psi(t) = \{0\}$, at $t > c$ one has $\partial\Psi(t) = \{1\}$ and at the kink $t = c$ one has $\partial\Psi(c) = [0, 1]$. The chain rule gives

$$\partial(\Psi \circ \phi)(x) \subseteq \partial\Psi(\phi(x)) \cdot \partial\phi(x) \subseteq [0, 1] \cdot \partial\phi(x) \subseteq \text{conv}(\{0_{X^*}\} \cup \partial\phi(x)),$$

which is precisely the max rule for ϕ and the constant function c , since (S4) holds for constants under (S1).

- (2). By (S1), we have $\partial\Psi(\phi(x)) = \partial^F\Psi(\phi(x))$. Since Ψ is nondecreasing, every $\alpha \in \partial\Psi(\phi(x))$ is nonnegative. For $\alpha > 0$, positive homogeneity (S6) gives

$$\partial(\alpha\phi)(x) \subseteq \alpha\partial\phi(x).$$

For $\alpha = 0$, the function $\alpha\phi$ is constant. Hence (S1) gives

$$\partial(0\phi)(x) = \{0_{X^*}\} = 0\partial\phi(x),$$

where the last equality uses $\partial\phi(x) \neq \emptyset$. Applying (S3) with $Y = \mathbb{R}$, $f = \phi$ and $\eta = \Psi$ therefore yields

$$\partial(\Psi \circ \phi)(x) \subseteq \bigcup_{\alpha \in \partial\Psi(\phi(x))} \partial(\alpha\phi)(x) \subseteq \bigcup_{\alpha \in \partial\Psi(\phi(x))} \alpha\partial\phi(x) = \partial\Psi(\phi(x)) \cdot \partial\phi(x).$$

This is (S8) for such outer functions Ψ . If the homogeneity inclusion is available for every nonzero $\alpha \in \mathbb{R}$, the same argument is valid without knowing the sign of the nonzero elements of $\partial\Psi(\phi(x))$, so the monotonicity assumption can be omitted.

- (3). Set $r := \phi - \psi$. Then r is locally constant around x . By (S4), $\partial r(x) = \{0_{X^*}\}$. Using (S5),

$$\partial\phi(x) = \partial(\psi + r)(x) \subseteq \partial\psi(x) + \partial r(x) = \partial\psi(x).$$

Interchanging ϕ and ψ gives the reverse inclusion. $\square$

2.3. Gerstewitz-Functional. We now introduce the Gerstewitz functional and collect several properties that will be used in the derivation of scalar optimality conditions below. The Gerstewitz functional is one of the central nonlinear scalarisation functionals in vector optimisation. It transforms vector-valued comparison statements into scalar inequalities while retaining information about the ordering set. In contrast to linear scalarisations, it is flexible enough to encode nonconvex ordering sets and nonconvex level-set structures. This makes it particularly useful in settings where convexity assumptions are too restrictive. A systematic treatment of this functional and its role in nonlinear scalarisation can be found, for instance, in Tammer and Weidner [17] or [6].

The following definition fixes our notation.

Definition 2.7. Let $\mathbb{V}$ be a linear space, $k \neq 0_{\mathbb{V}}$ and $D \subseteq \mathbb{V}$. We define the **Gerstewitz-functional**

$$\phi_{D,k} : \mathbb{V} \rightarrow \mathbb{R} \cup \{\pm\infty\}, \quad \phi_{D,k}(v) := \inf\{t \in \mathbb{R} \mid v \in -D + tk\}$$

Theorem 2.8 ([17, Theorem 4.4.4]). *Let $\mathbb{T}$ be a linear-topological space, $\emptyset \neq D \subset \mathbb{T}$ be a set and $k \in \mathbb{T} \setminus \{0_{\mathbb{T}}\}$ such that $\text{cl}(D) + \mathbb{R}_{>0}k \subseteq D$.*

- (1) $\phi_{D,k}$ is lower semicontinuous.
- (2) $\text{dom } \phi_{D,k} = \mathbb{R}k - D$ and for every $\lambda \in \mathbb{R}$

$$\text{lev}_{\leq}(\mathbb{T}, \phi_{D,k}, \lambda) = \lambda k - \text{cl}(D),$$

$$\phi_{D,k} = \phi_{\text{cl}(D),k}.$$

- (3) $\phi_{D,k}$ is convex if and only if $\text{cl}(D)$ is convex.
- (4) $\phi_{D,k}$ is positive homogeneous if and only if $\text{cl}(D)$ is a cone.

Remark 2.9. Let $\mathbb{T}$ be a real linear-topological space, let $\emptyset \neq D \subset \mathbb{T}$ be closed, let $C \subset \mathbb{T}$ be a proper, closed, solid convex cone, and assume the free-disposal property $D + C = D$. Then, for every $k \in \text{int}(C)$, the Gerstewitz functional $\phi_{D,k} : \mathbb{T} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ satisfies (see [8, Example 4.7])

$$\phi_{D,k} \in \mathcal{A}_i(-D, \mathbb{T}) \quad \text{for every } i \in \{1, 2, 4, 5, 6\}.$$

These elementary properties show why the Gerstewitz functional is well suited as a scalarisation device: its level sets are explicitly described by translates of the ordering set, and convexity or positive homogeneity of the scalar function can be read off directly from the geometry of the underlying set. For the multiplier rules below, we also need a subdifferential description.

Theorem 2.10 ([17, Theorem 4.10.8]). *Let $\mathbb{T}$ be a linear-topological space, let $D \subset \mathbb{T}$ be a proper convex cone and let $k \in \text{int}(D)$. Then $D + \mathbb{R}_{\geq 0}k \subseteq D$, and for every $y \in \mathbb{T}$ the Fenchel subdifferential $\partial^F \phi_{D,k}(y)$ is nonempty and*

$$\begin{aligned} \partial^F \phi_{D,k}(y) &= \{d^* \in D^{\geq} \mid d^*(k) = 1, d^*(y) = \phi_{D,k}(y)\}, \\ \partial^F \phi_{D,k}(0_{\mathbb{T}}) &= \{d^* \in D^{\geq} \mid d^*(k) = 1\}. \end{aligned}$$

In particular, $0_{\mathbb{T}^*} \notin \partial^F \phi_{D,k}(y)$ for every $y \in \mathbb{T}$. Hence every scalarising functional selected through the outer scalarisation chain rule is nonzero, so, when the theorem is applied to an objective mapping $f : X \rightarrow \mathbb{T}$, a subsequent stationarity condition of the form $0_{X^*} \in \partial(d^* \circ f)(\star)$ reflects the behaviour of f rather than arising from the degenerate scalarisation $0_{\mathbb{T}^*} \circ f$.

The explicit formula for $\partial^F \phi_{D,k}$ identifies the admissible scalar multipliers as positive continuous linear functionals normalised by $c^*(k) = 1$. The following proposition provides quantitative bounds on their norms, which will later yield corresponding bounds for the multipliers in the optimality conditions.

Proposition 2.11. *Let E be a normed space, let $D \subset E$ be a proper convex cone, and let $k \in \text{int}(D)$. Then, for every $z \in E$, every functional $d^* \in \partial^F \phi_{D,k}(z)$ satisfies*

$$\|k\|_E^{-1} \leq \|d^*\|_{E^*} \leq \text{dist}(k, \text{bd}(D))^{-1}. \quad (2.1)$$

In particular, for every $\epsilon > 0$ there exists $k \in \text{int}(D)$ such that

$$\|d^*\|_{E^*} < \epsilon \quad \text{for all } z \in E \text{ and all } d^* \in \partial^F \phi_{D,k}(z).$$

Proof. Since $k \in \text{int}(D)$, set $r := \text{dist}(k, \text{bd}(D)) > 0$. Let $z \in E$ and $d^* \in \partial^F \phi_{D,k}(z)$. Then $d^*(k) = 1$. Hence

$$1 = |d^*(k)| \leq \|d^*\|_{E^*} \|k\|_E,$$

which gives the lower bound $\|k\|_E^{-1} \leq \|d^*\|_{E^*}$.

If $h \in B_E^<(0_E; r)$, then $k \pm h \in \text{int}(D) \subseteq D$. Hence $1 \pm d^*(h) = d^*(k \pm h) \geq 0$, so $|d^*(h)| \leq 1$ for all $h \in B_E^<(0_E; r)$. For arbitrary $h \neq 0_E$ and $0 < \rho < r$, applying this estimate to $\rho h / \|h\|_E \in B_E^<(0_E; r)$ yields

$$|d^*(h)| \leq \rho^{-1} \|h\|_E.$$

Letting $\rho \nearrow r$ gives $|d^*(h)| \leq r^{-1} \|h\|_E$ for all $h \in E$, and therefore the upper bound in (2.1) follows.

Now let $\epsilon > 0$ and choose any $d \in \text{int}(D)$. Put $\delta := \text{dist}(d, \text{bd}(D)) > 0$. Since D is a cone, $\text{bd}(D) = \alpha \text{bd}(D)$ for every $\alpha > 0$, and thus

$$\text{dist}(\alpha d, \text{bd}(D)) = \inf_{b \in \text{bd}(D)} \|\alpha d - b\|_E = \alpha \inf_{b \in \text{bd}(D)} \|d - b\|_E = \alpha \delta.$$

Choosing $\alpha > 1/(\epsilon \delta)$ and setting $k := \alpha d$ gives $k \in \text{int}(D)$ and $\text{dist}(k, \text{bd}(D))^{-1} < \epsilon$. The upper bound then implies the asserted bound for all $z \in E$ and all $d^* \in \partial^F \phi_{D,k}(z)$. $\square$

We finally recall a Lipschitz estimate for the Gerstewitz functional.

Theorem 2.12 ([17, Lemma 4.6.16], [4, Lemma 2.4]). *Let Y be a normed space, let $C \subset Y$ be a closed convex cone and let $k \in \text{int}(C)$. Then $\phi_{C,k}$ is $\text{dist}(k, \text{bd}(C))^{-1}$ -Lipschitz continuous.*

We next recall the corresponding scalar characterisation of weak efficiency. It shows that weakly efficient points can be described as minimisers of a shifted Gerstewitz functional, thereby linking the vector-valued problem to a scalar minimality condition.

Theorem 2.13 ([17, Corollary 6.2.33]). *Let S be a set, let Y be a real linear-topological space, let $f : S \rightarrow Y$, let $C \subset Y$ be a proper convex cone and let $k \in \text{int}(C)$. Then*

$$\text{WEff}(f, S, C) = \left\{ x \in S \mid x \in \underset{x \in S}{\text{argmin}} \phi_{C,k}(f(x) - f(x)) \right\}.$$

Notice that $\phi_{C,k}(0_Y) = 0$.

The advantage of Theorem 2.13 is that it does not require convexity of f or S . The scalarised objective $\phi_{C,k}(f(\cdot) - f(x))$ is the composition of the vector mapping f with a scalar function that is Lipschitz continuous when Y is normed and C is closed. This is exactly the setting of property (S3).

2.4. Penalisation results for weak efficiency. We first recall generalised convexity concepts for functions that will be used to relate the solutions of the constrained and unconstrained problems.

Definition 2.14. Let X, Y be linear spaces, let $\mathcal{D} \subseteq X$, let $S \subseteq \mathcal{D}$ be convex and let $A \subseteq Y$. A function $f : \mathcal{D} \rightarrow Y$ is called **weakly A -quasiconvex** on S (in the sense of Jahn [11]) if

$$\forall x^1, x^2 \in S, x^1 \neq x^2, f(x^1) \leq_A f(x^2), \exists x^0 \in S \setminus \{x^2\}, \forall x \in [x^0, x^2] : f(x) \leq_A f(x^2).$$

Definition 2.15. Assume (A) and let $x^0 \in S$. The function f is called **C -upper semicontinuous** at x^0 if for any neighbourhood $V \subseteq Y$ of $f(x^0)$ there exists a neighbourhood $U \subseteq X$ of x^0 such that $f[U \cap S] \subseteq V - C$. Moreover, f is called **C -upper semicontinuous** on S if it has this property at every $x^0 \in S$.

Theorem 2.16 ([8, Theorem 3.4]). *Assume (A) and let f be weakly $\text{int}(C)$ -quasiconvex on X . Then*

$$\text{int}(S) \cap \text{WEff}(f, S, C) \subseteq \text{int}(S) \cap \text{WEff}(f, X, C), \quad (2.2)$$

$$S \cap \text{WEff}(f, X, C) \subseteq \text{WEff}(f, S, C) \subseteq (\text{int}(S) \cap \text{WEff}(f, X, C)) \cup (S \cap \text{bd}(S)), \quad (2.3)$$

$$\text{WEff}(f, S, C) \setminus \text{WEff}(f, X, C) \subseteq S \cap \text{bd}(S). \quad (2.4)$$

If additionally S is open, then $S \cap \text{WEff}(f, X, C) = \text{WEff}(f, S, C)$.

We now recall the vectorial penalisation results that connect the original constrained vector optimisation problem with unconstrained auxiliary problems. The idea is to encode the feasible set by means of a vector-valued penalisation term and then compare the efficient solution sets of the original objective with those of the penalised objective. We start with weak efficiency.

Theorem 2.17 ([8, Theorem 4.24]). *Assume $(A^\otimes)$. Then we have*

$$\mathcal{S} \cap \text{bd}(\mathcal{S}) \cap \text{WEff}(f, X, C) \subseteq \mathcal{S} \cap \text{bd}(\mathcal{S}) \cap \text{WEff}(f^\otimes, X, C \times \mathbb{R}_\geq), \quad (2.5)$$

$$\begin{aligned} & (\text{int}(\mathcal{S}) \cap \text{WEff}(f, X, C)) \cup (\mathcal{S} \cap \text{bd}(\mathcal{S}) \cap \text{WEff}(f^\otimes, X, C \times \mathbb{R}_\geq)) \\ &= (\mathcal{S} \cap \text{WEff}(f, X, C)) \cup (\mathcal{S} \cap \text{bd}(\mathcal{S}) \cap \text{WEff}(f^\otimes, X, C \times \mathbb{R}_\geq)). \end{aligned} \quad (2.6)$$

(1) *If ν additionally satisfies (A4), that is, $\nu \in \mathcal{A}_4(\mathcal{S}, X)$, then we get the barrier*

$$\mathcal{S} \cap \text{WEff}(f, X, C) \subseteq \text{WEff}(f, \mathcal{S}, C) \subseteq \mathcal{S} \cap \text{WEff}(f^\otimes, X, C \times \mathbb{R}_\geq). \quad (2.7)$$

(2) *Now assume that $\nu \in \mathcal{A}_1(\mathcal{S}, X) \cap \mathcal{A}_2(\mathcal{S}, X) \cap \mathcal{A}_7(\mathcal{S}, X)$ and that f is C -upper semicontinuous on $\mathcal{S}$. Then (2.7) holds and*

$$\begin{aligned} & \text{WEff}(f, \mathcal{S}, C) \\ & \supseteq (\text{int}(\mathcal{S}) \cap \text{WEff}(f, X, C)) \cup (\mathcal{S} \cap \text{bd}(\mathcal{S}) \cap \text{WEff}(f^\otimes, X, C \times \mathbb{R}_\geq)) \end{aligned} \quad (2.8)$$

$$= (\mathcal{S} \cap \text{WEff}(f, X, C)) \cup (\mathcal{S} \cap \text{bd}(\mathcal{S}) \cap \text{WEff}(f^\otimes, X, C \times \mathbb{R}_\geq)). \quad (2.9)$$

(3) *If instead $\nu \in \mathcal{A}_3(\mathcal{S}, X)$, then*

$$\mathcal{S} \cap \text{WEff}(f^\otimes, X, C \times \mathbb{R}_\geq) = \mathcal{S}.$$

(4) *If, in addition to $\nu \in \mathcal{A}_4(\mathcal{S}, X)$, we assume (2.4), then*

$$\begin{aligned} & \text{WEff}(f, \mathcal{S}, C) \\ & \subseteq (\text{int}(\mathcal{S}) \cap \text{WEff}(f, X, C)) \cup (\mathcal{S} \cap \text{bd}(\mathcal{S}) \cap \text{WEff}(f^\otimes, X, C \times \mathbb{R}_\geq)) \end{aligned} \quad (2.10)$$

$$= (\mathcal{S} \cap \text{WEff}(f, X, C)) \cup (\mathcal{S} \cap \text{bd}(\mathcal{S}) \cap \text{WEff}(f^\otimes, X, C \times \mathbb{R}_\geq)). \quad (2.11)$$

3. OPTIMALITY CONDITIONS

We now use the scalarisation and penalisation results recalled above to derive necessary optimality conditions for weakly efficient solutions of $\text{VOP}(f, \mathcal{S}, C)$. These conditions are formulated in terms of the general subdifferential introduced in Section 2.2.

Necessary optimality conditions for weakly efficient solutions of constrained vector optimisation problems in Banach spaces were also derived in [9, Theorem 4.1]. The results below allow objective mappings between linear-topological spaces and replace assumption (H2) from that theorem by the abstract Fermat condition $(S2_{\text{glob}}^\rightarrow)$. They also use the more flexible lower-level-set assumption $\nu \in \mathcal{A}_4(\mathcal{S}, X)$ for the penalisation function.

Theorem 3.1. *Assume $(A^\otimes)$ and $\text{int}(C) \neq \emptyset$, let $\mathfrak{x} \in \text{WEff}(f, \mathcal{S}, C)$ and let ∂ be a general subdifferential satisfying (S1), $(S2_{\text{glob}}^\rightarrow)$ and (S3).*

(1) *Let $\mathfrak{x} \in \text{int}(\mathcal{S})$ and either let f be weakly $\text{int}(C)$ -quasiconvex on X or let the subdifferential satisfy $(S2_{\text{loc}}^\rightarrow)$. Then, for every $k \in \text{int}(C)$ there exists $c^* \in C^\geq$ with $c^*(k) = 1$ such that*

$$0_{X^*} \in \partial(c^* \circ f)(\mathfrak{x}). \quad (3.1)$$

(2) *Let $\mathfrak{x} \in \mathcal{S}$ and assume that $\nu : X \rightarrow \mathbb{R}$ satisfies $\nu \in \mathcal{A}_4(\mathcal{S}, X)$. Then, for every $k := (k_0, t) \in \text{int}(C) \times \mathbb{R}_>$ there exist $u^* := (c^*, s^*) \in C^\geq \times \mathbb{R}_\geq$ with $u^*(k) = 1$ such that*

$$0_{X^*} \in \partial(u^* \circ f^\otimes)(\mathfrak{x}). \quad (3.2)$$

Proof. (1) For any $k \in \text{int}(C)$ we define

$$z : Y \rightarrow \mathbb{R}, \quad z(y) := \phi_{C,k}(y - f(\mathfrak{x})).$$

By [Theorem 2.13](#), $\mathfrak{x}$ is a minimum of $z \circ f$ on $\mathcal{S}$. If f is weakly $\text{int}(C)$ -quasiconvex on X , then [Theorem 2.16](#) gives $\text{int}(\mathcal{S}) \cap \text{WEff}(f, \mathcal{S}, C) \subseteq \text{WEff}(f, X, C)$. By [Theorem 2.13](#), $\mathfrak{x}$ is thus also a global minimum of $z \circ f$ on X . Combining this with $(\text{S2}_{\text{glob}}^{\rightarrow})$, or using $(\text{S2}_{\text{loc}}^{\rightarrow})$ directly, yields

$$0_{X^*} \in \partial(z \circ f)(\mathfrak{x}).$$

Since z is convex by [Theorem 2.8](#), the outer scalarisation chain rule [\(S3\)](#) gives

$$0_{X^*} \in \partial(z \circ f)(\mathfrak{x}) \subseteq \bigcup_{y^* \in \partial z(f(\mathfrak{x}))} \partial(y^* \circ f)(\mathfrak{x})$$

and by [\(S1\)](#) and [Theorem 2.10](#)

$$\partial z(f(\mathfrak{x})) = \partial \phi_{C,k}(0_Y) = \{c^* \in C^{\geq} \mid c^*(k) = 1\}.$$

(2) By [Theorem 2.17 \(1\)](#), $\text{WEff}(f, \mathcal{S}, C) \subseteq \mathcal{S} \cap \text{WEff}(f^{\otimes}, X, C \times \mathbb{R}_{\geq})$. For any $k = (k_0, t) \in \text{int}(C) \times \mathbb{R}_{>} = \text{int}(C \times \mathbb{R}_{\geq})$ we define

$$z : Y \times \mathbb{R} \rightarrow \mathbb{R}, \quad z(y, r) = \phi_{C \times \mathbb{R}_{\geq}, k}((y, r) - f^{\otimes}(\mathfrak{x})).$$

By [Theorem 2.13](#), $\mathfrak{x}$ is a minimum of $z \circ f^{\otimes}$ on X , so

$$0_{X^*} \in \partial(z \circ f^{\otimes})(\mathfrak{x}) \subseteq \bigcup_{u^* \in \partial z(f^{\otimes}(\mathfrak{x}))} \partial(u^* \circ f^{\otimes})(\mathfrak{x}).$$

Finally, [Theorem 2.10](#) gives $\partial z(f^{\otimes}(\mathfrak{x})) = \{u^* \in (C \times \mathbb{R}_{\geq})^{\geq} \mid u^*(k) = 1\}$. By [Proposition 2.2](#), $(C \times \mathbb{R}_{\geq})^{\geq} = C^{\geq} \times \mathbb{R}_{\geq}$, so $u^* = (c^*, s^*) \in C^{\geq} \times \mathbb{R}_{\geq}$ and $u^*(k) = c^*(k_0) + s^*t$. □

Remark 3.2. Since [VOP\(f, S, C\)](#) can be reformulated in terms of unconstrained problems, the optimality condition [\(3.2\)](#) contains no normal-cone term. Instead, feasibility information is incorporated through the subdifferential of the scalar penalisation function ν . This subdifferential is often explicit, or at least easier to estimate, in concrete settings such as distance-type penalties or cone-indicator constructions.

We next specialise [\(3.2\)](#) to several common descriptions of the feasible set.

Corollary 3.3. Assume $(A^{\otimes})$ and $\text{int}(C) \neq \emptyset$, and let ∂ be a general subdifferential satisfying [\(S1\)](#), $(\text{S2}_{\text{glob}}^{\rightarrow})$ and [\(S3\)](#) at $\mathfrak{x} \in \text{WEff}(f, \mathcal{S}, C)$. If $\nu \in \mathcal{A}_4(\mathcal{S}, X)$, then, for every $k := (k_0, t) \in \text{int}(C) \times \mathbb{R}_{>}$, there exists $u^* := (c^*, s^*) \in C^{\geq} \times \mathbb{R}_{\geq}$ with $u^*(k) = 1$ such that [\(3.2\)](#) holds. If $s^* = 0$, then

$$0_{X^*} \in \partial(c^* \circ f)(\mathfrak{x}). \tag{3.3}$$

If $s^* > 0$ and, additionally, the sum rule [\(S5\)](#) and the positive homogeneity property [\(S6\)](#) hold at $\mathfrak{x}$, then the condition can be written as the multiplier rule

$$0_{X^*} \in \partial(c^* \circ f + s^* \nu)(\mathfrak{x}) \subseteq \partial(c^* \circ f)(\mathfrak{x}) + s^* \partial \nu(\mathfrak{x}). \tag{3.4}$$

In particular, the following choices of ν under these additional assumptions lead to more explicit optimality conditions when $s^* > 0$; when $s^* = 0$, [\(3.3\)](#) applies without requiring the additional calculus rules.

(1) Let $g : X \rightarrow \mathbb{R}$, and suppose that the feasible set is given by one inequality:

$$\mathcal{S} = \{x \in X \mid g(x) \leq 0\}.$$

Then one obtains a KKT-type condition of the form

$$0_{X^*} \in \partial(c^* \circ f)(\mathfrak{x}) + s^* \partial g(\mathfrak{x}). \tag{3.5}$$

If, additionally, g is continuous at $\mathfrak{x}$ and [\(S4\)](#) is applicable, then

$$0_{X^*} \in \partial(c^* \circ f)(\mathfrak{x}), \quad \text{if } g(\mathfrak{x}) < 0. \tag{3.6}$$

(2) Let $g : X \rightarrow \mathbb{R}^m$, suppose that the feasible set is given by

$$\mathcal{S} = \left\{ x \in X \mid g(x) \leq_{\mathbb{R}_{\geq}^m} 0_{\mathbb{R}^m} \right\},$$

and define the active index set by $I(\mathfrak{x}) := \{i \in \{1, \dots, m\} \mid g_i(\mathfrak{x}) = 0\}$. If g is componentwise continuous at $\mathfrak{x}$, then under (S4) we have

$$0_{X^*} \in \partial(c^* \circ f)(\mathfrak{x}) \quad \text{if } I(\mathfrak{x}) = \emptyset. \quad (3.7)$$

Alternatively, assume that the max rule (S7) holds. Set

$$I_{\partial g}(\mathfrak{x}) := \{i \in \{1, \dots, m\} \mid \partial g_i(\mathfrak{x}) \neq \emptyset\}.$$

Then one obtains multipliers $\lambda = (\lambda_i)_{i \in I_{\partial g}(\mathfrak{x})} \in s^* \Delta_{|I_{\partial g}(\mathfrak{x})|}$ such that

$$0_{X^*} \in \partial(c^* \circ f)(\mathfrak{x}) + \sum_{i \in I_{\partial g}(\mathfrak{x})} \lambda_i \text{conv}(\partial g_i(\mathfrak{x})). \quad (3.8)$$

If $I(\mathfrak{x}) \neq \emptyset$, the max rule (S7) and (S4) hold, and g is componentwise continuous at $\mathfrak{x}$, then $I(\mathfrak{x}) \cap I_{\partial g}(\mathfrak{x}) \neq \emptyset$ and there exist multipliers $\lambda = (\lambda_i)_{i \in I(\mathfrak{x}) \cap I_{\partial g}(\mathfrak{x})} \in s^* \Delta_{|I(\mathfrak{x}) \cap I_{\partial g}(\mathfrak{x})|}$ such that

$$0_{X^*} \in \partial(c^* \circ f)(\mathfrak{x}) + \sum_{i \in I(\mathfrak{x}) \cap I_{\partial g}(\mathfrak{x})} \lambda_i \text{conv}(\partial g_i(\mathfrak{x})). \quad (3.9)$$

(3) Let $(Z, \|\cdot\|_Z)$ be a normed space, let $h : X \rightarrow Z$, and suppose that the feasible set is described by an equality constraint:

$$\mathcal{S} = \{x \in X \mid h(x) = 0_Z\}.$$

Then there exists $z^* \in \mathbb{B}_{Z^*}^{\leq}$ such that

$$0_{X^*} \in \partial(c^* \circ f)(\mathfrak{x}) + s^* \partial(z^* \circ h)(\mathfrak{x}). \quad (3.10)$$

(4) Let $(Z, \|\cdot\|_Z)$ be a normed space, let $g : X \rightarrow \mathbb{R}^m$ and $h : X \rightarrow Z$, and let

$$\mathcal{S} = \left\{ x \in X \mid g(x) \leq_{\mathbb{R}_{\geq}^m} 0_{\mathbb{R}^m}, h(x) = 0_Z \right\}.$$

Set $I(\mathfrak{x}) = \{i \in \{1, \dots, m\} \mid g_i(\mathfrak{x}) = 0\}$ and $I_{\partial g}(\mathfrak{x}) := \{i \in \{1, \dots, m\} \mid \partial g_i(\mathfrak{x}) \neq \emptyset\}$.

Let g be componentwise continuous at $\mathfrak{x}$, and assume that (S4) and the max rule (S7) hold. Then there exist multipliers $\lambda_i \geq 0$, $i \in I(\mathfrak{x}) \cap I_{\partial g}(\mathfrak{x})$, with $\sum_{i \in I(\mathfrak{x}) \cap I_{\partial g}(\mathfrak{x})} \lambda_i \leq s^*$, and $z^* \in \mathbb{B}_{Z^*}^{\leq}$ such that

$$0_{X^*} \in \partial(c^* \circ f)(\mathfrak{x}) + \sum_{i \in I(\mathfrak{x}) \cap I_{\partial g}(\mathfrak{x})} \lambda_i \text{conv}(\partial g_i(\mathfrak{x})) + s^* \partial(z^* \circ h)(\mathfrak{x}). \quad (3.11)$$

If $I(\mathfrak{x}) \cap I_{\partial g}(\mathfrak{x}) = \emptyset$, the empty sum in (3.11) is understood to be $\{0_{X^*}\}$.

Proof. If $s^* = 0$, then $c^* \circ f + s^* \nu = c^* \circ f$, so (3.2) directly gives (3.3). If $s^* > 0$ and the additional calculus rules hold, the sum rule (S5) and positive homogeneity (S6) give (3.4).

(1) $\mathfrak{x} \in \mathcal{S}$ implies $g(\mathfrak{x}) \leq 0$, so we consider the cases $g(\mathfrak{x}) < 0$ and $g(\mathfrak{x}) = 0$. If $g(\mathfrak{x}) = 0$, then the inequality is active at $\mathfrak{x}$. In this case we use $\nu = g$ as penalisation term, which satisfies $\nu \in \mathcal{A}_4(\mathcal{S}, X)$, and this implies a KKT-type condition of the form (3.5). If $g(\mathfrak{x}) < 0$, then the constraint is inactive. A natural choice for this case is

$$\nu(x) := g_+(x) := \max\{g(x), 0\}.$$

By [8, Example 4.13 and Theorem 4.9], we have $\nu \in \mathcal{A}_3(\mathcal{S}, X) \subseteq \mathcal{A}_4(\mathcal{S}, X)$. Continuity of g at $\mathfrak{x}$ implies that ν is locally constant around $\mathfrak{x}$. Hence (S4) gives $\partial \nu(\mathfrak{x}) = \{0_{X^*}\}$, and no additional multiplier term is produced. This proves (3.6).

(2) Since $\mathfrak{x} \in \mathcal{S}$, all values $g_i(\mathfrak{x})$ are nonpositive. If $I(\mathfrak{x}) = \emptyset$, one can take

$$\nu(x) := \max_{i=1,\dots,m} (g_i(x))_+ = \max \{0, g_1(x), \dots, g_m(x)\}.$$

By [8, Theorem 4.9], we have $\nu \in \mathcal{A}_3(\mathcal{S}, X) \subseteq \mathcal{A}_4(\mathcal{S}, X)$. Again, ν is locally constant around $\mathfrak{x}$, so (S4) yields (3.7).

The term

$$\nu(x) := \max_{i=1,\dots,m} g_i(x),$$

also satisfies $\nu \in \mathcal{A}_4(\mathcal{S}, X)$. The max rule (S7) gives

$$\partial\nu(\mathfrak{x}) \subseteq \text{conv} \left(\bigcup_{i=1}^m \partial g_i(\mathfrak{x}) \right).$$

For $i \in \{1, \dots, m\} \setminus I_{\partial g}(\mathfrak{x})$, the subdifferential $\partial g_i(\mathfrak{x})$ is empty and contributes nothing to that convex hull. Therefore, $I_{\partial g}(\mathfrak{x}) \neq \emptyset$. The standard description of convex hulls by finite convex combinations (see [10, Chapter A, Section 1.3, page 29]) yields

$$\text{conv} \left(\bigcup_{i=1}^m \partial g_i(\mathfrak{x}) \right) = \left\{ \sum_{i \in I_{\partial g}(\mathfrak{x})} \mu_i a_i \mid \mu \in \Delta_{|I_{\partial g}(\mathfrak{x})|}, a_i \in \text{conv}(\partial g_i(\mathfrak{x})) \text{ for } i \in I_{\partial g}(\mathfrak{x}) \right\}.$$

So there exists $\lambda \in s^* \cdot \Delta_{|I_{\partial g}(\mathfrak{x})|}$ satisfying (3.8).

If (S4) and (S7) are satisfied, we retain the previously introduced penalisation term and define

$$\eta(x) := \max_{i \in I(\mathfrak{x})} g_i(x).$$

Since $g_i(\mathfrak{x}) < 0$ for $i \notin I(\mathfrak{x})$, whereas $g_i(\mathfrak{x}) = 0$ for $i \in I(\mathfrak{x})$, continuity of the finitely many functions g_i implies that $\nu = \eta$ on a neighbourhood of $\mathfrak{x}$. Consequently, Proposition 2.6 (3) gives $\partial\nu(\mathfrak{x}) = \partial\eta(\mathfrak{x})$. The max rule therefore yields

$$\partial\nu(\mathfrak{x}) \subseteq \text{conv} \left(\bigcup_{i \in I(\mathfrak{x})} \partial g_i(\mathfrak{x}) \right).$$

Using the corresponding convex-hull representation then gives $\lambda \in s^* \Delta_{|I(\mathfrak{x}) \cap I_{\partial g}(\mathfrak{x})|}$ satisfying (3.9).

(3) We define

$$\nu(x) := \|h(x)\|_Z.$$

By [8, Theorem 4.9 or Remark 4.12], we have $\nu \in \mathcal{A}_3(\mathcal{S}, X) \subseteq \mathcal{A}_4(\mathcal{S}, X)$, so (3.4) is applicable. Since $h(\mathfrak{x}) = 0_Z$ and the norm is convex, (S1) gives

$$\partial \|0_Z\|_Z = \partial^F \|0_Z\|_Z = \{z^* \in Z^* \mid \forall y \in Z : z^*(y) \leq \|y\|_Z\} = \mathbb{B}_{Z^*}^{\leq}.$$

The chain rule (S3) therefore yields

$$\partial\nu(\mathfrak{x}) = \partial(\|\cdot\|_Z \circ h)(\mathfrak{x}) \subseteq \bigcup_{z^* \in \mathbb{B}_{Z^*}^{\leq}} \partial(z^* \circ h)(\mathfrak{x}).$$

Combining this inclusion with (3.4) gives (3.10).

(4) Define

$$\nu_+(x) := \max \{0, g_1(x), \dots, g_m(x)\} + \|h(x)\|_Z.$$

By [8, Remark 4.12], it follows that $\nu_+ \in \mathcal{A}_3(\mathcal{S}, X) \subseteq \mathcal{A}_4(\mathcal{S}, X)$. Therefore, (3.4) is applicable.

The result follows by the same argumentation as before, but since the maximum term in ν now contains zero, the convex hull representation by finite convex combinations can contain a multiplier for the zero functional resulting from the subdifferential of zero. $\square$

For an inactive inequality constraint, (3.6) recovers the stationarity condition (3.1) without requiring either weak $\text{int}(C)$ -quasiconvexity of f or the local Fermat rule ($\text{S2}_{\text{loc}}^{\rightarrow}$) used as alternatives in Theorem 3.1. Instead, the argument relies on the penalisation result, the global Fermat rule, and the local-constancy and calculus properties of the subdifferential.

Remark 3.4. Assume that in ($A^{\otimes}$) the spaces X and Y are normed spaces, and endow $Y \times \mathbb{R}$ with any norm compatible with the product topology, for example $\|(y, t)\|_{Y \times \mathbb{R}} := \|y\|_Y + |t|$. Fix directions $\bar{k}_C \in \text{int}(C)$ and $\bar{k}_{\otimes} \in \text{int}(C \times \mathbb{R}_{\geq})$. Since both sets are cones, positive rescaling gives

$$\begin{aligned} \text{dist}(\alpha \bar{k}_C, \text{bd}(C)) &= \alpha \text{dist}(\bar{k}_C, \text{bd}(C)), \\ \text{dist}(\alpha \bar{k}_{\otimes}, \text{bd}(C \times \mathbb{R}_{\geq})) &= \alpha \text{dist}(\bar{k}_{\otimes}, \text{bd}(C \times \mathbb{R}_{\geq})) \end{aligned}$$

for every $\alpha > 0$. Hence, for every $\epsilon > 0$, suitable positive rescalings k_C of $\bar{k}_C$ and $k_{\otimes}$ of $\bar{k}_{\otimes}$ can be chosen such that Proposition 2.11, applied with $D = C$ and $D = C \times \mathbb{R}_{\geq}$, respectively, yields multipliers in Theorem 3.1 satisfying

$$\|k_C\|_Y^{-1} \leq \|c^*\|_{Y^*} < \epsilon, \quad \|k_{\otimes}\|_{Y \times \mathbb{R}}^{-1} \leq \|u^*\|_{(Y \times \mathbb{R})^*} < \epsilon.$$

Thus the scalarisation multipliers c^* and $u^* = (c^*, s^*)$ can be chosen arbitrarily small in norm. Since the coordinate embeddings into $Y \times \mathbb{R}$ are continuous, there exist constants $M_Y, M_{\mathbb{R}} > 0$ such that

$$\|c^*\|_{Y^*} \leq M_Y \|u^*\|_{(Y \times \mathbb{R})^*}, \quad |s^*| \leq M_{\mathbb{R}} \|u^*\|_{(Y \times \mathbb{R})^*}.$$

Consequently, the coefficients s^* and λ_i occurring in ?? 3.3 can also be chosen arbitrarily small. This conclusion concerns the products involving the equality multiplier, such as $s^* z^*$, but not the normalised functional z^* itself.

Example 3.5 (An active constraint with empty Fenchel subdifferential). Let $X = Y = \mathbb{R}$, $C = \mathbb{R}_{\geq}$, $f(x) = -x$ and

$$g_1(x) = -x^2, \quad g_2(x) = x - 1, \quad g_3(x) = x.$$

Then

$$\mathcal{S} = \{x \in \mathbb{R} \mid g_1(x) \leq 0, g_2(x) \leq 0, g_3(x) \leq 0\} = (-\infty, 0],$$

and $\mathfrak{x} = 0$ is a global minimum of f on $\mathcal{S}$, hence $\mathfrak{x} \in \text{Weff}(f, \mathcal{S}, C)$. All three constraint functions are continuous and the active index set is $I(\mathfrak{x}) = \{1, 3\}$, whereas $g_2(\mathfrak{x}) = -1 < 0$.

Consider the Fenchel subdifferential. Directly from its definition,

$$\partial^F g_1(0) = \emptyset, \quad \partial^F g_2(0) = \partial^F g_3(0) = \{1\}.$$

Indeed, if $a \in \partial^F g_1(0)$, then $-x^2 \geq ax$ for every $x \in \mathbb{R}$, which is impossible. Thus the active concave constraint g_1 cannot receive a positive multiplier. The constraint g_2 is inactive and, moreover, $g_2(x) < g_3(x)$ for every $x \in \mathbb{R}$, so it does not contribute to the maximum penalty. Consequently,

$$\nu(x) := \max\{g_1(x), g_2(x), g_3(x)\} = \max\{-x^2, x\}.$$

For every $x^0 \in \mathcal{S}$, one has $\nu(x^0) \leq 0$, while $\nu(x) \leq 0$ implies $x \leq 0$. Hence $\text{lev}_{\leq}(\mathbb{R}, \nu, \nu(x^0)) \subseteq \mathcal{S}$, and therefore $\nu \in \mathcal{A}_4(\mathcal{S}, \mathbb{R})$. Furthermore,

$$\partial^F \nu(0) = \{1\} = \text{conv}(\partial^F g_1(0) \cup \partial^F g_3(0)),$$

so the required max-rule holds at $\mathfrak{x}$. To see the first equality, observe that $\nu(x) \geq x$ for every x , so $1 \in \partial^F \nu(0)$. Conversely, the inequalities from the subdifferential definition for $x > 0$ and $x \leq -1$ force every Fenchel subgradient at 0 to equal 1.

Choose $k = (1, 1)$ and $c^* = s^* = \frac{1}{2}$, so that $c^*(1) + s^* = 1$. The only possible positive active-constraint multiplier is therefore

$$\lambda_1 = 0, \quad \lambda_3 = \frac{1}{2},$$

and the optimality condition is satisfied because

$$0 \in \partial^F(c^* \circ f)(0) + \lambda_3 \partial^F g_3(0) = \left\{ -\frac{1}{2} \right\} + \left\{ \frac{1}{2} \right\}.$$

This example verifies the required Fenchel-subdifferential calculations and the max-rule directly. It does not assert that the Fenchel subdifferential satisfies the abstract local-constant and max rules for arbitrary nonconvex functions. Those properties generally require separate hypotheses.

Although the penalised formulation in [Theorem 3.1 \(3.2\)](#) avoids an explicit normal-cone term ([Remark 3.2](#)), such a term arises naturally if the abstract subdifferential is extended to proper extended-real-valued functions by setting $\partial\phi(x) = \emptyset$ whenever $\phi(x) = +\infty$, and if the corresponding Fermat and sum rules are assumed. The constraints can then be incorporated after scalarisation through the convex-analytic indicator function χ_S , leading to a condition involving $\partial(c^* \circ f + \chi_S)(x)$. For the standard subdifferentials, the subdifferential of the indicator function is the corresponding normal cone. This observation is conditional, since the abstract subdifferential and the penalisation function ν in the present framework are defined only for real-valued functions. Here we use the distance function instead. This keeps the scalar multiplier s^* from the penalised optimality condition visible and yields a slightly sharper form, in which only the unit-ball part of the normal cone enters, so that the impact of the penalisation term can be controlled. We first recall the convex normal cone and the corresponding subdifferential formula for the distance function.

Definition 3.6. Let $\mathbb{T}$ be a real topological vector space and let $A \subseteq \mathbb{T}$ be closed and convex. The convex normal-cone mapping $N_A : \mathbb{T} \rightrightarrows \mathbb{T}^*$ is defined by

$$N_A(x) := \begin{cases} \{x^* \in \mathbb{T}^* \mid x^*(y - x) \leq 0 \text{ for all } y \in A\}, & x \in A, \\ \emptyset, & x \notin A. \end{cases}$$

For a set $A \subseteq \mathbb{T}$ in a real topological vector space $\mathbb{T}$, define its convex-analytic indicator function $\chi_A : \mathbb{T} \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$\chi_A(x) := \begin{cases} 0, & x \in A, \\ +\infty, & x \notin A. \end{cases}$$

If A is convex, then $N_A = \partial^F \chi_A$.

Proposition 3.7 ([1, Section 3, Theorem 11]). *Let X be a normed space and let $S \neq \emptyset$ be closed and convex. Then, for every $x \in S$,*

$$\partial^F \text{dist}(\cdot, S)(x) = \mathbb{B}_{X^*}^{\leq} \cap N_S(x).$$

Proof. Let $x^* \in \partial^F \text{dist}(\cdot, S)(x)$. Since $\text{dist}(x, S) = 0$, the Fenchel subgradient inequality gives

$$x^*(y - x) \leq \text{dist}(y, S) \quad \text{for all } y \in X.$$

Taking $y \in S$ shows that $x^* \in N_S(x)$. Moreover, for every $v \in X$,

$$x^*(v) \leq \text{dist}(x + v, S) \leq \|v\|_X,$$

and applying the same estimate to $-v$ yields $|x^*(v)| \leq \|v\|_X$. Hence $\|x^*\|_{X^*} \leq 1$.

Conversely, let $x^* \in \mathbb{B}_{X^*}^{\leq} \cap N_S(x)$. For every $y \in X$ and $s \in S$,

$$x^*(y - x) = x^*(y - s) + x^*(s - x) \leq \|y - s\|_X.$$

Taking the infimum over $s \in S$ gives $x^*(y - x) \leq \text{dist}(y, S)$, so $x^* \in \partial^F \text{dist}(\cdot, S)(x)$. $\square$

Corollary 3.8. Assume $(A^\otimes)$ and $\text{int}(C) \neq \emptyset$, but let X be a normed space, let $\mathcal{S}$ be closed and convex, and let $\mathfrak{x} \in \text{Weff}(f, \mathcal{S}, C)$. Let ∂ be a general subdifferential satisfying (S1), $(S2_{\text{glob}}^\rightarrow)$, (S3), the sum rule (S5) and the positive homogeneity property (S6) at $\mathfrak{x}$. Then, for every $k = (k_0, t) \in \text{int}(C) \times \mathbb{R}_{>}$, there exist $c^* \in C^\geq$ and $s^* \in \mathbb{R}_{\geq}$ satisfying

$$c^*(k_0) + s^*t = 1$$

such that

$$0_{X^*} \in \partial(c^* \circ f)(\mathfrak{x}) + s^* \left(\mathbb{B}_{X^*}^\leq \cap N_{\mathcal{S}}(\mathfrak{x}) \right). \quad (3.12)$$

Proof. Define $\nu : X \rightarrow \mathbb{R}$ by $\nu(x) := \text{dist}(x, \mathcal{S})$. Since $\mathcal{S}$ is closed, $\nu(x) = 0$ holds exactly for $x \in \mathcal{S}$, and therefore $\nu \in \mathcal{A}_3(\mathcal{S}, X) \subseteq \mathcal{A}_4(\mathcal{S}, X)$. Thus Theorem 3.1 yields, for every $k = (k_0, t) \in \text{int}(C) \times \mathbb{R}_{>}$, multipliers $c^* \in C^\geq$ and $s^* \in \mathbb{R}_{\geq}$ with $c^*(k_0) + s^*t = 1$. If $s^* = 0$, then (3.2) directly gives

$$0_{X^*} \in \partial(c^* \circ f)(\mathfrak{x}).$$

If $s^* > 0$, the sum rule and positive homogeneity give

$$0_{X^*} \in \partial(c^* \circ f)(\mathfrak{x}) + s^* \partial \nu(\mathfrak{x}).$$

Because $\mathcal{S}$ is convex, the distance function ν is convex. Hence (S1) and Proposition 3.7 give

$$\partial \nu(\mathfrak{x}) = \partial^F \nu(\mathfrak{x}) = \mathbb{B}_{X^*}^\leq \cap N_{\mathcal{S}}(\mathfrak{x}).$$

Since $0_{X^*} \in \partial \nu(\mathfrak{x})$, the same displayed inclusion also holds when $s^* = 0$. Substitution of the subdifferential formula proves (3.12). $\square$

Remark 3.9. The convexity assumption in Proposition 3.7 is only needed for the elementary Fenchel-normal-cone formula used above. For closed nonconvex sets one can obtain analogous normal-cone inclusions if the abstract subdifferential is replaced by, or specialised to, a subdifferential with suitable distance-function calculus; for the Clarke subdifferential of the distance to a closed set, see [1, Section 3 and Theorem 11]. In contrast, the distance penalisation enforces membership in $\mathcal{S}$ exactly only when $\mathcal{S}$ is closed; for nonclosed feasible sets it naturally enforces membership in $\text{cl}(\mathcal{S})$, unless additional structure or a different penalisation is used.

Remark 3.10. Suppose that Y is a Banach space and that C is closed and convex. By Theorem 2.12, the Gerstewitz functionals associated with C on Y and with $C \times \mathbb{R}_{\geq}$ on $Y \times \mathbb{R}$ are Lipschitz continuous. Consequently, the scalar functions used in the proof of Theorem 3.1 are locally Lipschitz at $\mathfrak{x}$ whenever f is locally Lipschitz at $\mathfrak{x}$ in the interior case, respectively whenever $f^\otimes = (f, \nu)$ is locally Lipschitz at $\mathfrak{x}$ in the penalised case. Thus, in this Banach-space setting, it is sufficient for the Fermat rule $(S2_{\text{glob}}^\rightarrow)$ and the required chain rule to be available for locally Lipschitz functions and mappings.

The same observation applies to the special cases in ?? 3.3. It suffices to require that f and the relevant constraint mappings be locally Lipschitz at $\mathfrak{x}$: namely, g for a single inequality, every component g_i for finitely many inequalities, and h for an equality constraint. The corresponding penalisation terms are then locally Lipschitz because the positive part g_+ and finite maxima preserve local Lipschitz continuity, while the norm is Lipschitz and therefore $\|\cdot\|_Z \circ h$ is locally Lipschitz. Hence $f^\otimes$ is locally Lipschitz at $\mathfrak{x}$ in each case. If the outer scalarisation chain rule is restricted to a narrower function class, as discussed in Remark 2.4, membership of f and $f^\otimes$ in that class must still be verified separately.

This restriction is particularly natural for the Clarke subdifferential, which is commonly applied to locally Lipschitz functions. Fréchet, limiting/Mordukhovich, Dini and Ioffe approximate subdifferentials admit corresponding Fermat rules under weaker assumptions and therefore do not intrinsically require local Lipschitz continuity for this step.

The preceding results derive stationarity conditions that are necessary for weak efficiency. Under suitable sufficient Fermat properties and additional scalarisation or penalisation assumptions, these conditions can also be used in the converse direction. The following lemma collects several settings in which a stationary point is guaranteed to be weakly efficient.

Lemma 3.11. *Assume $(A^\otimes)$ and one of the following:*

- (1) $\mathfrak{x} \in \text{int}(\mathcal{S})$, f is weakly $\text{int}(C)$ -quasiconvex on some convex set $D \supseteq \mathcal{S}$, $(S2_{\text{loc}}^\leftarrow)$ and there exists $c^* \in C^\geq \setminus \{0_{Y^*}\}$ such that (3.1) holds.
- (2) $\mathfrak{x} \in \text{int}(\mathcal{S})$, $(S2_{\text{glob}}^\leftarrow)$ and there exists $c^* \in C^\geq \setminus \{0_{Y^*}\}$ such that (3.1) holds.
- (3) $\mathfrak{x} \in \mathcal{S}$, $(S2_{\text{glob}}^\leftarrow)$ and there exists $u^* := (c^*, s^*) \in (C^\geq \setminus \{0_{Y^*}\}) \times \mathbb{R}_\geq$ such that (3.2) and

$$\underset{x \in X}{\text{argmin}}(c^* \circ f + s^* \nu)(x) \subseteq \underset{x \in \mathcal{S}}{\text{argmin}}(c^* \circ f)(x). \quad (3.13)$$

- (4) $\mathfrak{x} \in \text{bd}(\mathcal{S})$, $(S2_{\text{glob}}^\leftarrow)$, f is C -upper semicontinuous on $\mathcal{S}$ and there exist $\nu \in \mathcal{A}_1(\mathcal{S}, X) \cap \mathcal{A}_2(\mathcal{S}, X) \cap \mathcal{A}_7(\mathcal{S}, X)$ and $u^* := (c^*, s^*) \in (C^\geq \times \mathbb{R}_\geq) \setminus \{(0_{Y^*}, 0)\}$ such that (3.2).

Then $\mathfrak{x} \in \text{WEff}(f, \mathcal{S}, C)$.

Proof. (1) Since $\mathfrak{x} \in \text{int}(\mathcal{S})$ and $0 \in \partial(c^* \circ f)(\mathfrak{x})$ for some $c^* \in C^\geq \setminus \{0_{Y^*}\}$, $(S2_{\text{loc}}^\leftarrow)$ implies that $\mathfrak{x}$ is a local minimum of $c^* \circ f$. Hence there exists a neighbourhood N of $\mathfrak{x}$ such that $\mathfrak{x}$ minimises $c^* \circ f$ over N . By [14, Chapter 4, Proposition 2.2] or [7, Proposition 5.1 and Remark 5.3] this implies $\mathfrak{x} \in \text{WEff}(f, \mathcal{S} \cap N, C)$. Since f is weakly $\text{int}(C)$ -quasiconvex on D , [8, Theorem 3.4] implies $\mathfrak{x} \in \text{WEff}(f, D, C) \cap \mathcal{S} \subseteq \text{WEff}(f, \mathcal{S}, C)$.

(2) By $(S2_{\text{glob}}^\leftarrow)$ and (3.1), $\mathfrak{x}$ is a global minimum of $c^* \circ f$. The same reasoning as in part (1) therefore gives $\mathfrak{x} \in \mathcal{S} \cap \text{WEff}(f, X, C) \subseteq \text{WEff}(f, \mathcal{S}, C)$.

(3) As before, $\mathfrak{x}$ is a global minimum of $u^* \circ f^\otimes = c^* \circ f + s^* \nu$. Hence (3.13) and [14, Chapter 4, Proposition 2.2] imply the assertion.

(4) Note that because of (A1) it is $\mathfrak{x} \in \text{bd}(\mathcal{S}) \subseteq \mathcal{S}$. The same argument as above gives $\mathfrak{x} \in \text{WEff}(f^\otimes, X, C \times \mathbb{R}_\geq)$. By Theorem 2.17 (2), this yields $\mathfrak{x} \in \text{WEff}(f, \mathcal{S}, C)$. $\square$

4. CONCLUSION

This paper combines nonlinear scalarisation, vectorial penalisation and abstract subdifferential calculus to derive necessary and, under additional assumptions, sufficient optimality conditions for weakly efficient solutions of constrained vector optimisation problems in real linear-topological spaces. The penalisation framework transfers the constrained problem to an unconstrained problem for the augmented objective (f, ν) , while the Gerstewitz functional produces nonzero, normalised scalar multipliers. Under a small collection of abstract Fermat and calculus rules, this leads to necessary stationarity conditions without requiring convexity of the objective mapping or of the feasible set.

The general conditions were specialised to single and finite systems of inequalities, equality constraints, mixed inequality–equality constraints and closed convex feasible sets. These specialisations yield KKT-type multiplier inclusions and, in the case of a convex feasible set, an explicit normal-cone formulation obtained from the distance penalisation. The axiomatic approach separates the scalarisation and penalisation arguments from the properties of any particular subdifferential, so the results can be transferred to concrete nonsmooth settings once the corresponding Fermat, sum and chain rules are available. This also clarifies which assumptions are needed for each form of the optimality conditions and provides a basis for future computational and theoretical developments.

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